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THE NATURAL CHARGE DISTRIBUTION AND CAPACITANCE OF A FINITE CONICAL SHELL

by

SAMUEL N. KARP

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Table of Contents

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ABSTRACT

The natural charge distribution for a conical cup has been obtained without approximation. Using spherical coordinates the potential u is expressed in the form $u = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu} A(\nu)P_{\nu-1}(\cos\Theta)d\nu$ for $0 < \Theta < \Theta_o$, and $u = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu}B(\nu)P_{\nu-1}(-\cos\Theta)d\nu$ for $\Theta_o < \Theta < \pi$. The two-part boundary value problem in r at $\Theta = \Theta_o$ then gives rise to a single equation between two functions of ν. This equation is solved by Wiener-Hopf techniques in the complex $\mu(=\nu-1/2)$ plane. This involves: (a) factorization of the product $2P_{-\frac{1}{2}+\mu}(\cos\Theta_o) \cdot P_{-\frac{1}{2}+\mu}(\cos\Theta_o)/\cos\pi\mu$ in the form $K^+(\mu) \cdot K^-(\mu)$, where $K^+(\mu)$ is regular in a right half-plane and $K^-(\mu)$ in a left half-plane; (b) the asymptotic forms of $K^+(\mu)$, $K^-(\mu)$. The necessary results are obtained by employing: (a) the fact that the zeros of $P_{-\frac{1}{2}+\mu}(\cos\Theta_o)$ considered as a function of μ are real, symmetric about the origin and asymptotically in arithmetic progression; (b) comparison of $K_+(\mu)$ with the gamma function of a suitable argument. From the knowledge of the natural charge distribution the capacitance of the conical cup is obtained as well as the behavior of the charge densities at the apex and the circular edge of the cup.

1. Introduction

1.1 General Considerations.

In a section of a previous report ([3]pp. 66–77) we have shown how the method of separation of variables in spherical coordinates, allied with Wiener-Hopf techniques, may be applied to the problem of determining the natural charge distribution on a circular disc. It was pointed out that the particular approach employed could be generalized to treat the problem involving a finite conical shell. Since this problem constitutes an addition to the finite list of boundary value problems whose solutions can be exhibited explicitly, it seems useful to provide a self contained exposition of the solution. This is the purpose of the present report.

We summarize briefly the ensuing argument. In section 1.2 we include, for reference, a sketch of the calculation for the disc which is the limiting form of the conical shell as the semi-vertex angle becomes a right angle. This simpler case serves as a guide in the more complicated considerations which follow. In section 2 the boundary value problem for the finite cone is stated and in section 3, using separation of variables in spherical coordinates we construct the following integral representation of the solution (see equations (3.6) and (3.7)

$$u(r,\theta) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu} A(\nu)P_{\nu-1}(\cos\theta)\, d\nu\,, \qquad \begin{matrix} 0 < r < \infty \\ 0 \le \theta \le \theta_0 \end{matrix}$$

(1.1)

$$u(r,\theta) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu} B(\nu)P_{\nu-1}(-\cos\theta)\, d\nu \qquad \begin{matrix} 0 < r < \infty \\ \theta_0 \le \theta \le \pi. \end{matrix}$$

Here θ_0 is the semi vertical angle at the apex of the cone, and the functions, $A(\nu)$ and $B(\nu)$, and the number $\delta = \delta(\theta_0)$ are to be determined. On subjecting these two integral expressions to the boundary conditions of the problem, we are led to the following integral equations for $A(\nu)$. (see equations (3.24) and (3.25)

$$0 = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu}\left\{A(\nu)P_{\nu-1}(\cos\theta_0) - \frac{u_0}{\nu}\right\}d\nu = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu}\, g_-(\nu)d\nu \qquad 0 < r < 1$$

(1.2)

$$0 = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu}\left\{\frac{1}{2\pi}\,\frac{\sin\pi\nu}{\sin\theta_0}\,\frac{A(\nu)}{P_{\nu-1}(\cos\theta_0)}\right\}d\nu = \int_{\delta-i\infty}^{\delta+i\infty} r^{-\nu}\,\phi_+(\nu)d\nu \qquad 1 < r < \infty\,,$$

where u_o is the potential of the conducting cone and where $\phi_+(\nu)$ and $g_-(\nu)$ represent the braced terms. Besides having the appropriate properties at infinity, $\phi_+(\nu)$ and $g_-(\nu)$ are assumed to be analytic in the regions Re $\nu \geq \delta(\theta_o)$ and Re $\nu \leq \delta(\theta_o)$, respectively. In this way the integral equations (1.2) are satisfied. Furthermore because of the relation of $\phi_+(\nu)$ to $g_-(\nu)$ through $A(\nu)$ we have, at least in the region of analyticity which $\phi_+(\nu)$ and $g_-(\nu)$ have in common, that

$$(1.3) \qquad K(\nu - \tfrac{1}{2}, \theta_o) \cdot \phi_+(\nu) - \frac{u_o}{\nu} = g_-(\nu)$$

where

$$(1.4) \qquad K(\nu - \tfrac{1}{2}, \theta_o) = \frac{2\pi \sin \theta_o}{\sin \pi\nu} P_{\nu-1}(\cos \theta_o)\, P_{\nu-1}(-\cos \theta_o).$$

It is at this point that Wiener-Hopf techniques become applicable. First of all it can be shown (section 5) that $K(\nu - \frac{1}{2}, \theta_o)$ can be written as

$$(1.5) \qquad K(\nu - \tfrac{1}{2}, \theta_o) = \sin \theta_o\, K^+(\nu - \tfrac{1}{2}, \theta_o) K^-(\nu - \tfrac{1}{2}, \theta_o) ,$$

where (1), $K^+(\nu - \frac{1}{2}, \theta_o)$ is regular for Re $\nu \geq \delta(\theta_o)$ and has an algebraic growth at infinity in this region and where (2), $K^-(\nu - \frac{1}{2}, \theta_o)$ is regular and without zeros in the region Re $\nu \leq \delta(\theta_o)$, and has an algebraic growth at infinity in this region. With these results in mind it is easily seen (section 6) that equation (1.3) can be rewritten, on setting $\mu = \nu - \frac{1}{2}$, as:

$$(1.6) \sin\theta_o K^+(\mu,\theta_o)\phi_+(\mu+\tfrac{1}{2}) - \frac{u_o}{(\mu+\frac{1}{2})K^-(-\frac{1}{2},\theta_o)} = \frac{g_-(\mu+\frac{1}{2})}{K^-(\mu,\theta_o)} - \frac{u_o}{\mu+\frac{1}{2}}\left[\frac{1}{K^-(-\frac{1}{2},\theta_o)} - \frac{1}{K^-(\mu,\theta_o)}\right].$$

On choosing $\delta(\theta_o)$ such that $0 < \delta(\theta_o) < 1$, it is then seen that the left hand side is the analytic for Re $\mu + \frac{1}{2} \geq \delta(\theta_o)$ and the right hand side is analytic for Re $\mu + \frac{1}{2} \leq \delta(\theta_o)$. Each side is therefore the analytic continuation of the other. Furthermore, it is shown, assuming the proper behavior of ϕ_+ and g_- at infinity in their regions of analyticity, that both sides tend to zero for large $|\mu|$. It then follows from Liouville's theorem that each side of (1.6) is identically zero. As a result $\phi_+(\mu+\frac{1}{2})$ can be found and hence so can $A(\nu)$ and $B(\nu)$, the latter being related to $A(\nu)$ be a simple formula (see equation 3.9).

In section 7 the integral expressions in equation (1.1) are evaluated by residues to obtain eigen-series representations of the potential in the various regions. The results obtained here are checked against the known results for the disc. In section 8 the capacitance of the cone is obtained. Again the result is checked with the known result for the disc. Finally, in section 9, we find the

behavior of the charge density at the apex and the circular edge of the cone.

Notes 1 to 4 are added to dispose of certain questions occurring in the course of the proof.

1.2 A Sketch of the Method Used for the Disc.

As already mentioned the solution of the disc problem given in [3], pp. 66-77, serves as a guide for the cone problem. We proceed here to outline the method used for the disc, highlighting those aspects of the procedure which are important for the cone.

Let the disc be located in the XY-plane as shown in Figure 1

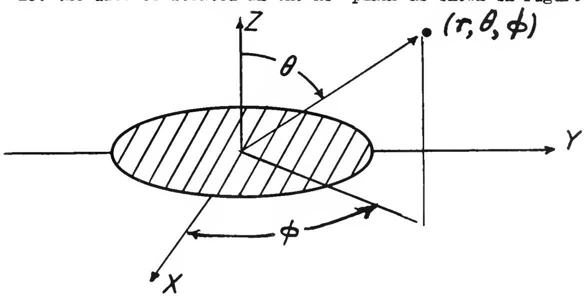

Figure 1

We require a solution to Laplace's equation, $\nabla^2 u = 0$, subject to the condition $u = u_0$ on the disc and that u together with ∇u be everywhere continuous except on the disc where ∇u may be discontinuous.* Introducing the customary spherical coordinates and their product solutions and taking into account the azimuthal symmetry, we assume, on allowing the separation constant $\nu - 1$ to be complex, that we may write

$$(1.7)\quad \begin{cases} u(r,\theta) = \dfrac{1}{2\pi i}\displaystyle\int_{\delta - i\infty}^{\delta + i\infty} r^{-\nu} \; A(\nu) P_{\nu-1}(\cos\theta)\, d\nu & 0 < r < \infty,\; 0 \leq \theta < \pi/2 \\[2ex] u(r,\theta) = \dfrac{1}{2\pi i}\displaystyle\int_{\delta - i\infty}^{\delta + i\infty} r^{-\nu} \; A(\nu) P_{\nu-1}(-\cos\theta)\, d\nu & 0 < r < \infty,\; \pi/2 \leq \theta \leq \pi \end{cases}$$

---

* We also require $u \to 0$ at infinity.

where $\delta = \delta(\pi/2)^*$, to be determined, is assumed positive and where $A(\nu)$ is also to be determined. These integral representations of $u(r,\theta)$ have been so designed as to avoid the singularities of the Legendre functions at $\theta = 0$ and $\theta = \pi$. Clearly $u(r,\theta)$ is continuous at $\theta_0 = \pi/2$ for $0 < r < \infty$. The fact that $u(r,\pi/2)=u_0$ for $0 < r < 1$ leads to the relation

$$(1.8)\qquad \int_{\delta - i\infty}^{\delta + i\infty} r^{-\nu} \left\{ A(\nu)P_{\nu-1}(0) - \frac{u_0}{\nu} \right\} d\nu = 0 , \qquad \text{for } 0 < r < 1,$$

where u_0 has been replaced by

$$(1.9)\qquad u_0 = \frac{u_0}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{r^{-\nu}}{\nu} d\nu \qquad 0 < r < 1 .$$

The continuity of $\frac{\partial u(r,\theta)}{\partial \theta}$ at $\theta = \pi/2$ for $1 < r < \infty$ leads to

$$(1.10)\qquad \int_{\delta - i\infty}^{\delta + i\infty} r^{-\nu} \left\{ A(\nu) \left. \frac{\partial P_{\nu-1}(\cos\theta)}{\partial\theta} \right|_{\theta = \pi/2} \right\} d\nu = 0 , \qquad r > 1.$$

Since it is known that

$$(1.11)\qquad P_{\nu-1}(0) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2} + \frac{\nu}{2})\Gamma(1 - \nu/2)} \qquad ([2], \text{ page } 63)$$

and

$$(1.12)\qquad \left. \frac{dP_{\nu-1}(\cos\theta)}{d(\cos\theta)} \right|_{\theta = \pi/2} = (\nu-1)P_{\nu-2}(0) \qquad ([2], \text{ page } 62) ,$$

it follows that

$$(1.13)\qquad \left. \frac{dP_{\nu-1}(\cos\theta)}{d(\cos\theta)} \right|_{\theta = \pi/2} = - \frac{2\sqrt{\pi}}{\Gamma(\frac{\nu}{2})\Gamma(\frac{1}{2} - \frac{\nu}{2})} .$$

---

* On page 1 we wrote $\delta = \delta(\theta_0)$ where θ_0 was the semi-vertical angle at the apex at the cone. For the case of the disc $\theta_0 = \pi/2$.

then, on calling the braced expressions in (1.8) and (1.10), $g(\nu)$, and $2\,\phi(\nu)$ respectively, we have

$$g(\nu) = \frac{\sqrt{\pi}\,A(\nu)}{\Gamma(\frac{1}{2}+\frac{\nu}{2})\,\Gamma(1-\frac{\nu}{2})} - \frac{u_o}{\nu} \tag{1.14}$$

and

$$\phi(\nu) = \frac{\sqrt{\pi}\,A(\nu)}{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{1}{2}-\frac{\nu}{2})} \tag{1.15}$$

In terms of these functions, equations (1.8) and (1.10) may be written as

$$\left\{\begin{array}{ll} \displaystyle\int_{\delta - i\infty}^{\delta + i\infty} r^{-\nu}\, g(\nu)\, d\nu = 0 & 0 < r < 1 \\[2ex] \displaystyle\int_{\delta - i\infty}^{\delta + i\infty} r^{-\nu}\, \phi(\nu)\, d\nu = 0 & r > 1 \end{array}\right. \tag{1.16}$$

Assuming that,

1) $g(\nu)$ is analytic in the region $\mathrm{Re}\,\nu \leq \delta\,(\pi/2)$ and that as $|\nu| \to \infty$, $g(\nu) \sim \nu^{-p}$, with $p > 0$, in this region,

2) $\phi(\nu)$ is analytic in the region $\mathrm{Re}\,\nu \geq \delta(\pi/2)$ and that as $|\nu| \to \infty$, $\phi(\nu) \sim \nu^{-q}$ with $q > 0$, in this region, then the equations in (1.16) will be satisfied. To emphasize the fact that these assumptions have been made we shall write $g(\nu)$ as $g_-(\nu)$ and $\phi(\nu)$ as $\phi_+(\nu)$. Now because of the form of $\phi(\nu)$ in equation (1.15) we can write the expression for $g_-(\nu)$ in (1.14), at least for $\mathrm{Re}\,\nu = \delta(\pi/2)$, in the following form

$$g_-(\nu) = -\frac{u_o}{\nu} + \phi_+(\nu)\, K^-(\nu - \tfrac{1}{2}, \pi/2)\, K^+(\nu - \tfrac{1}{2}, \pi/2)\,, \tag{1.17}$$

where

$$\left\{\begin{array}{l} K^-(\nu - \frac{1}{2}, \pi/2) = \dfrac{\Gamma(\frac{1}{2} - \frac{\nu}{2})}{\Gamma(1 - \frac{\nu}{2})} \\[3ex] K^+(\nu - \frac{1}{2}, \pi/2) = \dfrac{\Gamma(\nu/2)}{\Gamma(\frac{1}{2} + \frac{\nu}{2})}\,. \end{array}\right. \quad \text{(compare with (1.5))} \tag{1.18}$$

It is important to note that,

1) $K^+(\nu - \frac{1}{2}, \pi/2)$ is regular in the region Re $\nu > 0$, and is asymptotically $\nu^{-1/2}$ * as $|\nu|$ approaches infinity in this region.

2) $K^-(\nu - \frac{1}{2}, \pi/2)$ is regular in the region Re $\nu < 1$, has no zeroes and is asymptotically $(-\nu)^{-1/2}$ as $|\nu|$ approaches infinity in this region.

Now, (1.17) can be rewritten as:

(1.19)
$$\frac{g_-(\nu)}{K^-(\nu - \frac{1}{2}, \pi/2)} + \frac{u_o}{\nu}\left[\frac{1}{K^-(\nu - \frac{1}{2}, \pi/2)} - \frac{1}{K^-(-\frac{1}{2}, \pi/2)}\right] =$$

$$\phi_+(\nu)K^+(\nu - \frac{1}{2}, \pi,/2) - \frac{u_o}{\nu}\frac{1}{K^-(-\frac{1}{2}, \pi/2)} ,$$

where the left hand side is regular for Re ν less than the first positive root of $K^-(\nu - \frac{1}{2}, \pi/2)$, [in this case, at $\nu = 2$, from (1.18)] and where the right hand side is regular for Re $\nu \geq \delta(\pi/2) > 0$. Thus if we choose $\delta(\pi/2)$ such that $\delta(\pi/2) < 2$, then each side is the analytic continuation of the other. Furthermore if we assume merely that $g_-(\nu) \sim \nu^{-p}$ where $p > \frac{1}{2}$ and $\phi_+(\nu) \sim \nu^{+q}$ where $q < \frac{1}{2}$ ***, then applying Stirling's formula it is seen that both sides vanish at infinity. Invoking Liouville's theorem, we can conclude

(1.20)
$$\phi_+(\nu) = \frac{u_o}{\sqrt{\pi}\,\nu\,K^+(\nu - \frac{1}{2}, \pi/2)}$$

and hence by (1.15)

$$A(\nu) = \frac{u_o}{\pi}\frac{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{1}{2} - \frac{\nu}{2})}{\nu\,K^+(\nu - \frac{1}{2}, \pi/2)} .$$

---

* To see this it is only necessary to apply Stirling's formula.

** It is actually necessary to take $\delta(\frac{\pi}{2}) < 1$, in order that we may be able to solve for $g_-(\nu)$. Note that $\left(K^-(\nu - \frac{1}{2}, \pi/2)\right)^{-1}$ is zero at $\nu = 1$.

*** Note that conditions 1 and 2 on page 5 are thereby satisfied.

Thus,

$$(1.21)\quad A(\nu) = \frac{u_o}{\pi} \; \frac{\Gamma(\frac{1}{2} - \frac{\nu}{2})\,\Gamma(\frac{1}{2} + \frac{\nu}{2})}{\nu}$$

(from (1.18)),

or equivalently,

$$(1.22)\quad A(\nu) = \frac{u_o}{\nu \cos(\pi\nu/2)}$$

For the case of the cone, as already suggested by equations (1.3) and (1.5), we shall arrive at an equation completely analogous to equation (1.17) except $\sin\theta_o \cdot K^+(\nu - \frac{1}{2}, \theta_o) \cdot K^-(\nu - \frac{1}{2}, \theta_o)$ will appear in the place of $K^+(\nu - \frac{1}{2}, \pi/2) \cdot K^-(\nu - \frac{1}{2}, \pi/2)$. From this equation on, all general considerations go through as in the disc case except it is much harder to factor the coefficient of (1.17) into $\sin\theta_o\, K^+(\nu - \frac{1}{2}, \theta_o)\, K^-(\nu - \frac{1}{2}, \theta_o)$ where the K's have the properties 1) and 2) mentioned on page 2. We proceed now to the detailed treatment of the cone problem.

2. Statement of the Problem

Let C be the finite right conical shell, shown in Figure 2 below.

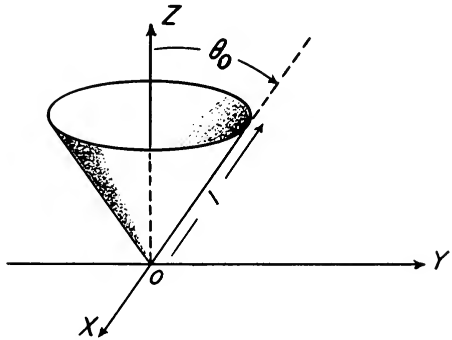

Figure 2

We assume that the generators of the cone make an angle $\theta = \theta_0$, $(0<\theta_0<\pi)$ with the z axis and, for convenience, that the slant height of the cone is unity. In the spherical polar coordinates

$$x = r \sin \theta \cos \phi \tag{2.1}$$
$$y = r \sin \theta \sin \phi$$
$$z = r \cos \theta,$$

the surface of the cone is given by,

$$x = r \sin \theta_0 \cos \phi \tag{2.2}$$
$$y = r \sin \theta_0 \sin \phi$$
$$z = r \cos \theta_0 ,$$

where $0 \leq r \leq 1$ and $0 < \theta_0 < \pi$.

For C we propose to solve the following boundary value problem: Find a function $u(r,\theta)$,* satisfying Laplace's equation which is subject to the conditions

I. (a) $u(r,\theta_o) = u_o$, for $0 < r < 1$, where u_o is a constant,

(b) $u(r,\theta)$ is continuous for $0 < r < \infty$ and $0 \leq \theta \leq \pi$

(c) ∇u is continuous for all r and θ except possibly on cone; $\theta = \theta_o$; $0 \leq r \leq 1$

(d) $\lim_{r \to \infty} u(r,\theta) = 0$ uniformly in θ, $0 \leq \theta \leq \pi$.

Writing Laplace's equations in spherical polar coordinates and taking into account the fact that u is independent of the azimuthal angle, ϕ, we get:

$$\text{II.} \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u(r,\theta)}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u(r,\theta)}{\partial\theta}\right) = 0 .$$

The physical significance of the problem is the following: a charge, Q, say, is placed on a conductor of the shape in question, and allowed to distribute itself freely. An electrostatic field of potential $u(r,\theta)$ is thereby set up with the conducting cone as part of the equipotential surface $u = u_o$ in this field. The relation between u_o and Q can be found after the problem is solved, for if ω is the charge density on the cone, C, then the surface integral $\int_C \omega \, ds$ over the cone is equal to Q. Since ω can be computed in terms of u_o, at the completion of the problem, we can arrive at the relation between Q and u_o. Alternatively, it will be possible to express $u(r,\theta)$ in the form

$$u(r,\theta) = \frac{A(\theta)}{R} + \frac{B(\theta)}{R^2} + \ldots$$

on a sphere of large radius centered at the origin. Employing Gauss' theorem which states that the surface integral $\int_S \frac{\partial u}{\partial n} dS$ over a surface enclosing a charge distribution is $4\pi Q$, it is easily seen that

$$(2.3) \qquad \lim_{R \to \infty} \int_S \frac{\partial u}{\partial R} \, dS = -\int A(\theta) \, d\Omega = 4\pi Q$$

where $d\Omega$ is the solid angle subtended by a portion of S at the origin. Since $A(\theta)$ will involve u_o we again can find the relationship between Q and u_o.

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* Because of the symmetry of our problem about the z-axis we assume that u is independent of the azimuthal angle ϕ of the spherical polar coordinate system (2.1)

Incidentally, once this is done, we can calculate the capacitance of the finite cone.

3. Representation of the Solution.

Employing the method of separation of variables we set $u(r,\theta)=R(r)S(\theta)$. By the usual method we arrive at the two equations

$$(3.1) \qquad r^2 \frac{d^2R}{dr^2} + 2r \frac{dR}{dr} - \rho(\rho+1) R = 0,$$

$$(3.2) \qquad \frac{d}{d\theta}\left(\sin\theta \frac{dS}{d\theta}\right) + \rho(\rho+1) S = 0, \qquad 0 < \theta < \pi .$$

Here ρ is permitted to be complex. Thus a solution of equation II is

$$(3.3) \qquad V_\rho(r,\theta) = \frac{1}{r^{\rho+1}} P_\rho(\cos\theta)$$

where $r^{-(\rho+1)}$ satisfies (3.1) and where $P_\rho(\cos\theta)$ satisfies (3.2). $P_\rho(\cos\theta)$, for $-1 < \cos\theta \leq 1$, is defined for all complex values of ρ by an integral expression, the so-called Schläfli integral expression, [[1] page 191, equation 15]. $P_\rho(\cos\theta)$, given in this manner is an entire function of ρ for $-1 < \cos\theta \leq 1$.

Now if $P_\rho(\cos\theta)$ satisfies (3.2) then so does $P_\rho(-\cos\theta)$. This statement follows, for instance, from the equation

$$(3.4) \quad P_\rho(-\cos\theta) = \cos\rho\pi P_\rho(\cos\theta) + \frac{2\sin\rho\pi}{\pi} Q_\rho(\cos\theta),$$

where $\rho \neq 0, \pm 1, \pm 2, \ldots$ ([2] page 56, Eq.7).

Here $Q_\rho(\cos\theta)$ is the Legendre function of the second kind, and $P_\rho(\cos\theta)$ is the Legendre function of the first kind. For our purposes it is sufficient to know that $Q_\rho(\cos\theta)$, $-1 < \cos\theta < 1$, is a solution of (3.2) which is linearly independent of $P_\rho(\cos\theta)$. From (3.4) we can then assert that $P_\rho(-\cos\theta)$ is a solution of (3.2) which is not linearly dependent on $P_\rho(\cos\theta)$. Thus, as another solution of Laplace's equation we have, provided $\rho \neq 0, \pm 1, \pm 2, \ldots$

$$(3.5) \qquad W_\rho(r,\theta) = \frac{1}{r^{\rho+1}} P_\rho(-\cos\theta) , \qquad 0 < \theta \leq \pi$$

Note that $P_\rho(\cos\theta)$ is regular except at $\theta = \pi$, and $P_\rho(-\cos\theta)$ is regular except at $\theta = 0$.

Letting $\rho + 1 = \nu$, we proceed to construct, formally, a solution of the boundary value problem by superposing the $V_{\nu-1}(r,\theta)$ and the $W_{\nu-1}(r,\theta)$ in the following manner:**

(3.6) $$u(r,\theta) = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu} A(\nu)\, P_{\nu-1}(\cos\theta)\, d\nu \qquad \begin{matrix} 0 \leq \theta \leq \theta_0 \\ 0 < r < \infty \end{matrix}$$

(3.7) $$u(r,\theta) = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu}\, B(\nu)\, P_{\nu-1}(-\cos\theta)\, d\nu \qquad \begin{matrix} \theta_0 \leq \theta \leq \pi \\ 0 < r < \infty \end{matrix}$$

Note that δ will in general be a function at θ_0, the semi-vertex angle of the cone. The use of $P_{\nu-1}(\cos\theta)$ in (3.6) and $P_{\nu-1}(-\cos\theta)$ in (3.7) avoids singularities in the Legendre function at $\theta = 0$, $\theta = \pi$, respectively, while admitting a possible discontinuity at $\theta = \theta_0$. Here $A(\nu)$ and $B(\nu)$ are unknown functions which we propose to find by imposing on (3.6) and (3.7) the conditions given in I and II.

First of all the condition that $\lim_{r\to\infty} u(r,\theta) = 0$, uniformly in θ, suggests that we take $\mathrm{Re}\,\nu = \delta(\theta_0) > 0$; for on factoring out $r^{-\delta}$ from the above integrals, the desired effect is seen to obtain provided the remaining integral expressions are bounded in θ. In order that $u(r,\theta)$ be continuous across the infinite cone $\theta = \theta_0$, $0 < r < \infty$ we must have

(3.8) $$0 = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu}\left[A(\nu)\, P_{\nu-1}(\cos\theta_0) - B(\nu)\, P_{\nu-1}(-\cos\theta_0)\right] d\nu$$

for $0 < r < \infty$.

Inverting this expression by the Mellin-transformation* we get formally that

(3.9) $$A(\nu)\, P_{\nu-1}(\cos\theta_0) = B(\nu)\, P_{\nu-1}(-\cos\theta_0)$$

Incidentally, it will be noted that when $\theta_0 = \pi/2$, that is, the cone degenerates to a disc, $P_{\nu-1}(\cos\pi/2) = P_{\nu-1}(0) = P(-\cos\pi/2)$. As a result $A(\nu) = B(\nu)$ and it follows, when we are dealing with the special case of the conducting disc,

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* See [2], page 137.

** Once and for all we shall designate the contour of integration which is parallel to the Im ν axis at Re $\nu = \delta$, by C_δ.

that we may write

$$u(r,\theta) = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu} A(\nu)\, P_{\nu-1}(|\cos\theta|)\, d\nu \quad .$$

The problem of the conducting disc has already been treated in this manner (see equation (1.7), page 3). In most of what follows we shall employ the methods used in solving the disc problem as a guide in obtaining results for the cone and as a check on the final results.

Now condition I-c tells us that $\nabla u(r,\theta)$ must be continuous for all points not on the cone, that is $\nabla u(r,\theta)$ must be continuous across the surface $\theta = \theta_0$, $1 < r < \infty$. Assuming we may interchange the operation of integration and differentiation with respect to r and θ*, it is first of all apparent by an inspection of (3.6) and (3.7) that $\frac{\partial u}{\partial r}(r,\theta)$ is continuous for all r, $0 < r < \infty$. Hence it is only necessary to discuss $\frac{\partial}{\partial\theta} u(r,\theta)$. Now for $0 < r < \infty$

$$(3.10) \qquad \frac{\partial}{\partial\theta} u(r,\theta)\Big|_{\theta_0-\epsilon} = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu} A(\nu) \frac{d}{d\theta} P_{\nu-1}(\cos\theta)\Big|_{\theta=\theta_0-\epsilon} d\nu$$

$$(3.11) \qquad \frac{\partial}{\partial\theta} u(r,\theta)\Big|_{\theta=\theta_0+\epsilon} = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu} B(\nu) \frac{d}{d\theta} P_{\nu-1}(-\cos\theta)\Big|_{\theta=\theta_0+\epsilon} d\nu$$

The jump in $\frac{\partial u}{\partial\theta}$ as the cone $\theta = \theta_0$ is crossed, is the limit of the difference of corresponding sides of (3.10) and (3.11), so that the condition that $\frac{\partial}{\partial\theta} u(r,\theta)$ be continuous across $\theta = \theta_0$ for $1 < r < \infty$, results in the following relation:

$$(3.12) \qquad 0 = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu}\left\{A(\nu)\frac{d}{d\theta} P_{\nu-1}(\cos\theta) - B(\nu)\frac{d}{d\theta} P_{\nu-1}(-\cos\theta)\right\}\Big|_{\theta=\theta_0} d\nu$$

for $1 < r < \infty$.

If we set

$$(3.13)\quad \phi(\nu) = \frac{\pi}{4}\left\{A(\nu)\frac{d}{d\theta} P_{\nu-1}(\cos\theta) - B(\nu)\frac{d}{d\theta} P_{\nu-1}(-\cos\theta)\right\}\Big|_{\theta=\theta_0} ,$$

then at least on $\operatorname{Re}\nu = \delta(\theta_0)$, we have from (3.9), formally, that,

*Actually interchanging integration with differentiation here, for the contour C_δ, does not appear permissible; the procedure here should be regarded as purely heuristic. The solution which we obtain later will be subjected to an independent check. (cf. Note 3.)

$$(3.14)\quad \begin{cases} \phi(\nu) = \frac{\pi}{4}\ \frac{B(\nu)}{P_{\nu-1}(\cos\theta_o)} \left\{ \left(P_{\nu-1}(-\cos\theta)\frac{d}{d\theta}P_{\nu-1}(\cos\theta) - P_{\nu-1}(\cos\theta)\frac{d}{d\theta}P_{\nu-1}(-\cos\theta) \right)\Big|_{\theta=\theta_o} \right\} \\ \text{or} \\ \phi(\nu) = \frac{\pi}{4}\ \frac{A(\nu)}{P_{\nu-1}(-\cos\theta_o)} \left\{ \left(P_{\nu-1}(-\cos\theta)\frac{d}{d\theta}P_{\nu-1}(\cos\theta) - P_{\nu-1}(\cos\theta)\frac{d}{d\theta}P_{\nu-1}(-\cos\theta) \right)\Big|_{\theta=\theta_o} \right\}. \end{cases}$$

These expressions for $\phi(\nu)$ can be further simplified, for letting, $x = \cos\theta$, we have $\frac{d}{d\theta} = -\sin\theta\frac{d}{dx} = -\sqrt{1-x^2}\ \frac{d}{dx}$. Now the expression (3.14) involves the Wronskian of $P_{\nu-1}(\cos\theta)$ and $P_{\nu-1}(-\cos\theta)$. Denoting the expressions in the braces in (3.14) by $W\left[P_{\nu-1}(\cos\theta_o), P_{\nu-1}(-\cos\theta_o)\right]$, we get in terms of x,

$$(3.15)\qquad W\left[P_{\nu-1}(\cos\theta_o), P_{\nu-1}(-\cos\theta_o)\right] = -\sqrt{1-x^2}\ W\left[P_{\nu-1}(x), P_{\nu-1}(-x)\right]$$

But equation (3.4), in our present notations, tells us that

$$(3.16)\quad P_{\nu-1}(-x) = \cos(\nu-1)\pi\ P_{\nu-1}(x) + \sin(\nu-1)\pi\ Q_{\nu-1}(x) \qquad \nu \neq 0, \pm1, \pm2, \ldots$$

Thus on substituting into (3.15), we get

$$(3.17)\quad W\left[P_{\nu-1}(\cos\theta_o), P_{\nu-1}(-\cos\theta_o)\right] = -\frac{2\sqrt{1-x^2}}{\pi}\sin(\nu-1)\pi\ W\left[P_{\nu-1}(x), Q_{\nu-1}(x)\right].$$

Now,

$$(3.18)\qquad W\left[P_{\nu-1}(x), Q_{\nu-1}(x)\right] = \frac{1}{1-x^2}\ , \quad ([2]\ \text{page } 63)\ ,$$

so that we have finally,

$$(3.19)\qquad W\left[P_{\nu-1}(\cos\theta_o), P_{\nu-1}(-\cos\theta_o)\right] = \frac{2\sin\pi\nu}{\pi\sqrt{1-x^2}} = \frac{2\sin\pi\nu}{\pi\sin\theta_o}\ .$$

The expression for $\phi(\nu)$ then becomes,

$$(3.20)\qquad \phi(\nu) = \begin{cases} \frac{1}{2\pi}\ \frac{\sin\pi\nu}{\sin\theta_o}\ \frac{B(\nu)}{P_{\nu-1}(\cos\theta_o)}\ , \text{ or} \\ \frac{1}{2\pi}\ \frac{\sin\pi\nu}{\sin\theta_o}\ \frac{A(\nu)}{P_{\nu-1}(-\cos\theta_o)} \end{cases}$$

Thus far conditions I (b), (c) and (d) have been imposed on the integral expressions given in (3.6) and (3.7). Condition I (a) imposed on (3.6) yields

$$(3.21) \qquad u_o = \frac{1}{2\pi i} \int_{C_\delta} r^{-\nu} A(\nu) P_{\nu-1}(\cos \theta_o) \, d\nu \ , \qquad 0 < r < 1 \ .$$

However,

$$u_o = \frac{u_o}{2\pi i} \int_{C_\delta} \frac{r^{-\nu}}{\nu} \, d\nu, \qquad 0 < r < 1,$$

as can be easily seen by

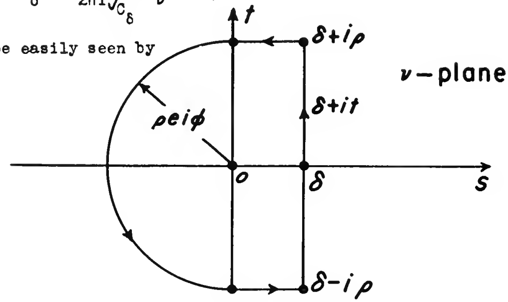

Figure 3

evaluating the integral in the right hand side by residues, over the path shown in figure 3, and taking the limit as $\rho \to \infty$. Setting

$$(3.23) \qquad g(\nu) = A(\nu) P_{\nu-1}(\cos \theta_o) - \frac{u_o}{\nu}$$

we have by virtue of (3.21) and (3.23)

$$(3.24) \qquad \int_{C_\delta} r^{-\nu} g(\nu) \, d\nu = 0 \qquad 0 < r < 1$$

Also, from (3.12) and (3.13) we have

$$(3.25) \qquad \int_{C_\delta} r^{-\nu} \phi(\nu) \, d\nu = 0 \qquad 1 < r < \infty \ .$$

We are thus led to the problem of solving the pair of integral equations, involving $A(\nu)$ and $B(\nu)$, given in (3.24) and (3.25)

4. Solution of the Integral Equations of Section 3

Starting with equation (3.25) and assuming merely** that (1), $\phi(\delta + \rho e^{i\phi})$ is analytic for $\rho > \rho_0$ and that (2), $\lim_{\rho \to \infty} \phi(\delta + \rho e^{i\phi}) \to 0$ uniformly for ϕ in $\pi/2 \leq \phi \leq -\pi/2$*, then by an argument similar to that used in proving Jordan's Lemma, it can be shown that

$$(4.1) \quad \lim_{\rho \to \infty} \int_{C_\rho} r^{-(\delta + \rho e^{i\phi})} \phi(\delta + \rho e^{i\phi}) \rho e^{i\phi} \, d\phi \to 0$$

uniformly in ϕ for $-\pi/2 \leq \phi \leq \pi/2$, provided $1 < r < \infty$. Thus the contour in equation (3.25) is equivalent to the contour shown in figure 4 below, as $\rho \to \infty$.

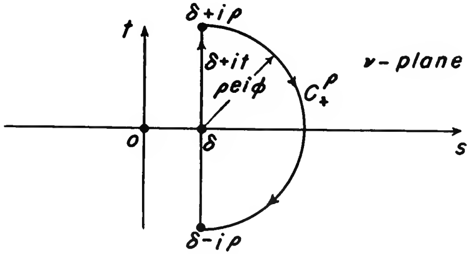

Figure 4

Now if we further assume $\phi(\delta + \rho e^{i\phi})$ is analytic for all ρ, $0 \leq \rho < \infty$ and $-\pi/2 \leq \phi \leq \pi/2$, then by the method of residues we see that equation (3.25) is satisfied. To remind ourselves of this assumption we shall write $\phi(\nu) = \phi_+(\nu)$, where the plus sign indicates regularity of $\phi(\nu)$ in the region $\mathrm{Re}\,\nu \geq \delta(\Theta_0)$. Having thus specified the behavior of $\phi_+(\nu)$ in this region, we know we can continue $\phi_+(\nu)$ somewhat into the region to the left of the line $\mathrm{Re}\,\nu = \delta(\Theta_0)$. So much for equation (3.25) at least for the present.

Turning now to solving (3.24) it can be shown that, in the limit, the contour in equation (3.24) is equivalent to the contour shown in figure 3, provided only that (1), $g(\nu) \to 0$ uniformly in ϕ for $\pi/2 \leq \phi \leq 3/2\pi$ and for $0 \leq \mathrm{Re}\,\nu \leq \delta(\Theta_0)$, as $\rho \to \infty$ and that (2) $g(\nu)$ is regular for $\mathrm{Re}\,\nu \leq \delta(\Theta_0)$. Thus if we assume $g(\nu)$

---

*We shall assume $\phi(\nu)$ has a suitable algebraic growth at ∞ for instance $\phi(\nu) \sim \nu^{-q}$ for some number $q > 0$. Actually q will turn out to be 1/2.

** It is hoped that no confusion will arise from the use of ϕ in the expression $\phi(\nu)$, and its use as a geometrical quantity in the combination $\rho e^{i\phi}$. (see Figure 4.)

to be regular in the entire region Re $\nu \leq \delta(\theta_o)$ and of algebraic growth at infinity in this region (i.e., $g(\nu)$ behaves like ν^{-p} at infinity where p is some positive number), then from the theory of residues equation (3.24) is satisfied. In analogy to the notation we have used for $\phi(\nu)$ we write $g(\nu)=g_-(\nu)$ to emphasize the fact that $g(\nu)$ is analytic in the region Re $\nu \leq \delta(\theta_o)$. Now since the region of analyticity includes the line Re $\nu = \delta(\theta_o)$ it is possible by analytic continuation to extend $g_-(\nu)$ somewhat into the region Re $\nu \geq \delta(\theta_o)$. Then at least for the region of analyticity which $\phi_+(\nu)$ and $g_-(\nu)$ have in common we can write, using (3.20) and (3.23)

$$\text{(4.2)} \qquad \frac{2\pi \sin\theta_o}{\sin \pi\nu} P_{\nu-1}(-\cos\theta_o)\, P_{\nu-1}(\cos\theta_o)\, \phi_+(\nu) - \frac{u_o}{\nu} = g_-(\nu).$$

In order to motivate the following section let us consider the coefficient of $\phi_+(\nu)$ in equation (4.2). Calling this coefficient $K(\nu - 1/2, \theta_o)$, we have

$$\text{(4.3)} \qquad K(\nu - 1/2, \theta_o) = \frac{2\pi \sin\theta_o}{\sin \pi\nu} P_{\nu-1}(\cos\theta_o)\, P_{\nu-1}(-\cos\theta_o)\ .$$

Setting $\theta_o = \pi/2$ this expression should reduce to the corresponding expression in the disc problem. From [2] page 63 we know that

$$\text{(4.4)} \qquad P_{\nu-1}(0) = \frac{\sqrt{\pi}}{\Gamma(1/2 + \nu/2)\,\Gamma(1 - \nu/2)}\ .$$

Also, $\sin \pi\nu = \sin 2\pi(\nu/2) = 2\sin\pi(\nu/2) \cdot \sin\pi(1/2 - \nu/2)$

$$= \frac{2\pi^2}{\Gamma(\nu/2)\Gamma(1-\nu/2)\,\Gamma(1/2-\nu/2)\Gamma(1/2+\nu/2)}\ .$$

On substituting into (4.3) we get the expression

$$\text{(4.5)} \qquad K(\nu - 1/2, \pi/2) = \left[\frac{\Gamma(\nu/2)}{\Gamma(1/2 + \nu/2)}\right]\left[\frac{\Gamma(1/2 - \nu/2)}{\Gamma(1 - \nu/2)}\right],$$

which checks with the coefficient at $\phi_+(\nu)$ in equation (1.17), page 5.

Now in treating the disc problem it was pointed out that the first factor is regular and has no zeros for Re $\nu > 0$, and secondly, that it has an algebraic growth in this region as $|\nu| \to \infty$. Similarly the second factor is

analytic, has no zeros, and has an algebraic growth as $|\nu| \to \infty$ for all ν such that Re $\nu < \delta(\pi/2)$*. Considering now the more general expression for $K^{+}(\nu-1/2,\theta_0)$ given in equations (4.3) we should like to effect a corresponding factorization in this $K(\nu - 1/2, \theta_0)$. Let us for reasons of symmetry rewrite (4.3) as,

$$
\begin{aligned}
K(\mu,\theta_0) &= \frac{2\pi\sin\theta_0}{\sin\pi(\mu+1/2)}\, P_{-1/2+\mu}(\cos\theta_0)\, P_{-1/2+\mu}(-\cos\theta_0) \\
&= 2\sin\theta_0\, \Gamma(\mu+1/2)\,\Gamma(1/2-\mu)\, P_{-1/2+\mu}(\cos\theta_0) P_{-1/2+\mu}(-\cos\theta_0),
\end{aligned}
\tag{4.6}
$$

where we have set $\nu = 1/2 + \mu$.

In view of what has been said above we want to write $K(\mu,\theta_0)$ in the form

$$K(\mu,\theta_0) = \sin\theta_0\, K^{+}(\mu,\theta_0)\, K^{-}(\mu,\theta_0) \tag{4.7}$$

where, (1), $K^{+}(\mu,\theta_0)$ is regular for Re $(\mu + 1/2) \geq \delta\ (\theta_0)$ and has an algebraic growth at ∞ for μ in this region, and where (2), $K^{-}(\mu,\theta_0)$ is regular and without zeros in the region Re$(\mu + 1/2) \leq \delta(\theta_0)$, and has an algebraic growth at ∞ for μ in this region. It will be shown in the next section that such a factorization is possible and that for Re $\mu \geq 0$, $K^{+}(\mu,\theta_0) \sim \text{const}\ \mu^{-1/2}$ as $|\mu| \to \infty$ whereas $K^{-}(\mu,\theta_0) \sim \text{const}.\ (-\mu)^{-1/2}$, Re $\mu \leq 0$. This result will be obtained in three steps. First of all, the behavior of the zeros of $P_{-1/2+\mu}(\pm\cos\theta_0)$ in equation (4.6) will be investigated. Secondly, $P_{-1/2+\mu}(\cos\theta_0)$ and $P_{1/2+\mu}(-\cos\theta_0)$ will be written in the form

$$
\begin{aligned}
P_{-1/2+\mu}(\cos\theta) &= P_{-1/2}(\cos\theta_0)\, k^{+}(\mu,\theta_0)\, k^{-}(\mu,\theta_0) \\
P_{-1/2+\mu}(\cos(\pi-\theta_0)) &= P_{-1/2}(\cos(\pi-\theta_0)) k^{+}(\mu,\pi-\theta_0)\, k^{-}(\mu,\pi-\theta_0)
\end{aligned}
\tag{4.8}
$$

where the k^{+}'s and k^{-}'s will be certain infinite products having zeros in the half planes Re$\mu \leq 0$ and Re$\mu \geq 0$, respectively. Thirdly, $K^{+}(\mu,\theta_0)$ and $K^{-}(\mu,\theta_0)$

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* see page 6

will be defined as

$$(4.9)\quad \begin{cases} K^{+}(\mu,\theta_0) = \sqrt{2}\, P_{-1/2}(\cos\theta_0)\, e^{f(u)}\, \Gamma(1/2+\mu) k^{+}(\mu,\theta_0) k^{+}(\mu,\pi-\theta_0) \\ K^{-}(\mu,\theta_0) = \sqrt{2}\, P_{-1/2}(\cos(\pi-\theta_0)) e^{-f(u)} \Gamma(1/2-\mu) k_1^{-}(u,\theta_0) k^{-}(\mu,\pi-\theta_0) \end{cases}$$

where $f(u)$ is so determined that the growths of K^{+} and K^{-} are algebraic in the proper regions.

The next section will be subdivided into three subsections corresponding to each of these steps.

5. The Expansion at $P_{-1/2+\mu}(\pm x)$ as an Infinite Product

5.1 The Zeros of $P_{-1/2+\mu}(\pm x)$

We begin by listing some properties of Legendre functions. From [1], pages 188-9, we have $P_{\nu}(z) = P_{-(\nu+1)}(z)$ for all z in the circle $|z-1| < 2$ and so in particular for $-1 < x \le 1$ we have $P_{\nu}(x) = P_{-(\nu+1)}(x)$. On setting $\nu = \mu - 1/2$ we get

$$(5.1)\quad \begin{gathered} P_{\mu-1/2}(x) = P_{-\mu-1/2}(x)\,, \\ \text{or} \\ P_{\mu-1/2}(\cos\theta_0) = P_{-\mu-1/2}(\cos\theta_0)\,, \end{gathered}$$

where $x = \cos\theta_0$. We can therefore assert, that for fixed θ_0, $P_{\mu-1/2}(x)$ is an even function of μ. From this result we can immediately conclude that if $\mu(\theta_0)$ is a zero of $P_{\mu-1/2}(x)$, then $-\mu(\theta_0)$ is also a zero of $P_{\mu-1/2}(x)$.

Now it is known ([2], page 70) that all the zeros of $P_{\mu-1/2}(x)$ are real and simple. From page 71 of the same reference, for ζ real, positive and large compared to 1, and $0 < \epsilon < \theta_0 \le \pi-\epsilon$, we have,

$$(5.2)\quad P_{\zeta}(\cos\theta_0) = \sqrt{\frac{2}{\pi\zeta\sin\theta_0}}\cos\left[(\zeta+1/2)\theta_0 - \pi/4\right]\left\{1 + O(\zeta^{-3/2})\right\}.$$

It follows that, for very large positive ζ, the zeros of $P_{\zeta}(\cos\theta_0)$ are given asymptotically by *

$$(5.3)\quad \zeta \sim -1/2 + \frac{\pi}{\theta_0}(n+3/4) + \frac{C(\theta_0)}{n}, \quad 0 < \epsilon \le \theta_0 \le \pi - \epsilon .$$

---

*Actually neglecting terms of $O(n^{-2})$, ζ can be given more precisely by

$$\zeta = -1/2 + \frac{\pi}{\theta_0}(n+3/4) + \frac{\cot\theta_0}{8\theta_0\left[-\frac{1}{2} + \frac{\pi}{\theta_0}(n+3/4)\right]} + O\left(\frac{1}{n^2}\right)$$

This result is from a paper entitled "On the Numerical Calculation of the Roots of $P_n^m(\mu) = 0$", by Bholanath Pal in the Bulletin of the Calcutta Mathematical Society Vol. IX, March 1919, Page 88.

, Now letting $\varsigma = -1/2 + \mu$, the zeros of $P_{\mu-1/2}(\cos\theta_o)$ are then asymptotically

$$(5.4) \quad \mu \sim \frac{\pi}{\theta_o}(n + 3/4) + \frac{c(\theta_o)}{n}, \quad 0 < \epsilon \leq \theta_o \leq \pi - \epsilon$$

It must, however, not be assumed that the mth positive zero for large m is given by setting n = m in (5.4). To see this, note that for $\theta_o = \pi/2$ the footnote on page 18 tells us that the zeros of $P_{-1/2+\mu}(0)$ are $2(n+3/4)$ up to terms order n^{-2}. Now we also know from [2], page 63, on setting $\varsigma = -1/2+\mu$, that

$$(5.5) \quad P(0) = \frac{\sqrt{\pi}}{\Gamma(1+\varsigma/2)\Gamma(1/2-\varsigma/2)} = \frac{\sqrt{\pi}}{\Gamma(3/4+\mu/2)\Gamma(3/4-\mu/2)} .$$

Thus for this case all the positive zeros of $P_{-1/2+\mu}(0)$ are given exactly by

$$\mu'_n = 2(n+3/4) \qquad n = 0, 1, 2, \ldots,$$

thus μ'_o is the first zero of $P_{-1/2+}(0)$.

Now on setting n = m-1 and setting $\mu'_{m-1} = \mu_m$, we have

$$\mu_m = 2(m-1/4) \qquad m = 1, 2, \ldots,$$

This equation suggest all the positive zeros of $P_{-1/2+\mu}(\cos\theta_o)$ are given for large m by

$$(5.6) \quad \mu_m(\theta_o) \sim \frac{\pi}{\theta_o}(m-1/4) + \frac{c(\theta_o)}{m}, \qquad 0 < \epsilon \leq \theta_o \leq \pi - \epsilon ,$$

where the mth zero is $\mu_m(\theta_o)$. This last fact is actually the case, as can be readily verified graphically ([4] page 108, figure 58)*.

Turning now to $P_{-1/2+\mu}(-\cos\theta_o)$, we set $\pi - \theta_o = \chi_o$. It follows that the large positive zeros $P_{-1/2+\mu}(-\cos\theta_o) = P_{-1/2+\mu}(\cos\chi_o)$ are given approximately by

$$(5.7) \quad \mu_m(\chi_o) \sim \frac{\pi}{\chi_o}(m-1/4) + \frac{c(\chi_o)}{m} \qquad 0 < \epsilon \leq \chi_o \leq \pi - \epsilon .$$

---

* The author is indebted to Professor W. Magnus who has kindly supplied an analytic proof of the fact that $\mu_m(\theta_o)$ is a continuous function of θ_o and is monotonically decreasing as θ_o goes from 0 to π. Combining this result with the information on the zeros of $P_\varsigma(0)$ and the fact that the zeros of $P_{-1/2+\mu}(x)$ are simple, we have an analytic proof that the mth positive zero is $\mu_m(\theta_o)$.

Because, as already mentioned above, $P_{-1/2+\mu}(\pm x)$ is an even function of μ for fixed x, the negative zeros of $P_{-1/2+\mu}(\pm x)$ are obtained from these positive ones by reflecting about the imaginary axis in the μ plane.

5.2 The Factorization of $P_{-1/2+\mu}(\pm x)$ - The Asymptotic Behavior of the Factors

It can be shown*, as suggested by (5.5) that we may write

$$(5.8)\quad P_{-1/2+\mu}(\cos\theta_o) = P_{-1/2}(\cos\theta_o)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_m(\theta_o)}\right)e^{\frac{-\mu}{\mu_m(\theta_o)}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_m(\theta_o)}\right)e^{\frac{\mu}{\mu_m(\theta_o)}}$$

and

$$(5.9)\quad P_{-1/2+\mu}(-\cos\theta_o) = P_{-1/2}(\cos\chi_o)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_m(\chi_o)}\right)e^{\frac{-\mu}{\mu_m(\chi_o)}}\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_m(\chi_o)}\right)e^{\frac{\mu}{\mu_m(\chi_o)}}$$

Let us call the first infinite product appearing in (5.8) $k^+(\mu,\theta_o)$ and the second $k^-(\mu,\theta_o)$. Similarly in (5.2) we write, $k^+(\mu,\chi_o)$, and $k^-(\mu,\chi_o)$, for the first and second infinite products respectively. For our future purposes it will be necessary to know the asymptotic behavior of $k^+(\mu,\theta_o)$ and $k^+(\mu,\chi_o)$ in the half plane $\mathrm{Re}(\mu+1/2)\geq\delta(\theta_o)$ and of $k^-(\mu,\theta_o)$, $k^-(\mu,\chi_o)$ in the half plane $\mathrm{Re}\ (\mu+1/2)\leq\delta(\theta_o)$.

Confining our attention to $k^+(\mu,\theta_o)$ we shall compare the growth of $k^+(\mu,\theta_o)$, for large μ, with the growth of the infinite product $L(\mu) = \prod_{m=1}^{\infty}\left(1+\frac{\theta_o\mu}{\pi}/(m-\frac{1}{4})\right)\cdot e^{-\frac{\theta_o\mu}{\pi}/(m-\frac{1}{4})}$. In turn, L(u), can be related to the Gamma function. In fact we have

$$(5.10)\qquad \frac{\Gamma(a+1)}{\Gamma(z+a+1)}\cdot e^{z\psi(a+1)} = \prod_{m=1}^{\infty}\left(1+\frac{z}{m+a}\right)e^{\frac{-z}{m+a}} \quad ,$$

where $\psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$. Taking $z = \frac{\theta_o\mu}{\pi}$ and $a = -1/4$, we get

$$(5.11)\qquad L(\mu) = \frac{\Gamma(3/4)}{\Gamma\left(\frac{\theta_o\mu}{\pi}+3/4\right)}\cdot e^{\frac{\theta_o\mu}{\pi}\psi(3/4)} = \prod_{m=1}^{\infty}\left(1+\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}\right)e^{\frac{-\mu}{\frac{\pi}{\theta_o}(m-1/4)}} ,$$

which gives the desired relation of L (u) with the Gamma function. Now form the ratio $R(\mu)$, of $k^+(\mu,\theta_o)$ and L(u). Thus

---

* See Note 1.

$$(5.12)\quad R(\mu) = \frac{k^+(\mu,\theta_o)}{L(\mu)} = \frac{\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_m(\theta_o)}\right) e^{-\frac{\mu}{\mu_m(\theta_o)}}}{\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}\right) e^{-\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}}}$$

Taking the logarithm of both sides and adding corresponding terms we get

$$\log R(\mu) = \sum_{m=1}^{\infty} \log \left[\left(\frac{1+\frac{\mu}{\mu_m(\theta_o)}}{1+\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}}\right) e^{\mu\left(\frac{1}{\frac{\pi}{\theta_o}(m-1/4)} - \frac{1}{\mu_m(\theta_o)}\right)}\right]$$

$$= \sum_{m=1}^{\infty} \log \left[\frac{\left(1+\frac{\mu}{\mu_m(\theta_o)}\right)}{\left(1+\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}\right)}\right] + \left[\sum_{m=1}^{\infty}\left(\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)} - \frac{\mu}{\mu_m(\theta_o)}\right)\right] ,$$

since each series converges separately for all μ, as a consequence of the asymptotic property of $\mu_m(\theta)$ (See (5.4)). It follows

$$(5.13)\quad R(\mu) = \prod_{m=1}^{\infty}\left(\frac{1+\frac{\mu}{\mu_m(\theta_o)}}{1+\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}}\right) \cdot e^{\frac{\theta_o\mu}{\pi}\sum_{m=1}^{\infty}\left(\frac{1}{m-1/4} - \frac{1}{\frac{\theta_o}{\pi}\mu_m(\theta_o)}\right)} .$$

For large $|\mu|$, in the region $|\arg \mu| \geq \pi/2$, $R(\mu)$ becomes therefore*

$$R(\mu) \sim \left[\prod_{m=1}^{\infty}\left\{\frac{(m-1/4)}{\frac{\theta_o}{\pi}\mu_m(\theta_o)}\right\}\right] \cdot e^{\left\{\frac{\theta_o}{\pi}\sum_{m=1}^{\infty}\left(\frac{1}{m-1/4} - \frac{1}{\frac{\theta_o}{\pi}\mu_m(\theta_o)}\right)\right\}}$$

or

$$(5.14)\qquad R(\mu) \sim A(\theta_o)\, e^{\frac{\theta_o}{\pi}B(\theta_o)} ,$$

where $A(\theta_o)$ denotes the infinite product and $B(\theta_o)$ the infinite sum.** From Equations (5.11) and (5.12) we have then

* See Note 2

** When $\theta_o = \pi/2$, $A(\theta_o)=1$ and $B(\theta_o)=0$ so that $R(\mu)\sim 1$ as $|\mu| \longrightarrow \infty$.

$$(5.15)\quad k^{+}(\mu,\theta_o) \sim \frac{\Gamma(3/4)}{\Gamma(\frac{\theta_o}{\pi}\mu+3/4)} A(\theta_o)\, e^{\frac{\mu\theta_o}{\pi}\left[B(\theta_o)+\psi(3/4)\right]}, \quad \mathrm{Re}\,\mu \geq 0$$

A similar treatment of $k^{+}(\mu,\chi_o)$, $k^{-}(\mu,\theta_o)$, $k^{-}(\mu,\chi_o)$ yields:

$$k^{+}(\mu,\chi_o) \sim \frac{\Gamma(3/4)}{\Gamma(\frac{\chi_o}{\pi}\mu+3/4)} \cdot A(\chi_o)\, e^{\frac{\mu\chi_o}{\pi}\left[B(\chi_o)+\psi(3/4)\right]}, \quad \mathrm{Re}\,\mu \geq 0$$

$$k^{-}(\mu,\theta_o) \sim \frac{\Gamma(3/4)}{\Gamma(3/4-\frac{\theta_o}{\pi}\mu)} \cdot A(\theta_o)\, e^{\frac{-\mu\theta_o}{\pi}\left[B(\theta_o)+\psi(3/4)\right]}, \quad \mathrm{Re}\,\mu \leq 0$$

$$k^{-}(\mu,\chi_o) \sim \frac{\Gamma(3/4)}{\Gamma(3/4-\frac{\chi_o}{\pi}\mu)} \cdot A(\chi_o)\, e^{\frac{-\mu\chi_o}{\pi}\left[B(\chi_o)+\psi(3/4)\right]}, \quad \mathrm{Re}\,\mu \leq 0$$

5.3 The Factorization of the Coefficient $K(\mu,\theta_o)$.

From equation (4.6) on page 17 we know

$$(5.16)\quad K(\mu,\theta_o) = 2\sin\theta_o\, \Gamma(\mu+1/2)\, \Gamma(1/2-\mu) P_{-1/2+\mu}(\cos\theta_o) P_{-1/2+\mu}(\cos\chi_o)$$

From (5.8) and (5.9) we have

$$(5.17)\quad K(\mu,\theta_o) = \sin\theta_o \left\{\sqrt{2}P_{-1/2}(\cos\theta_o)\Gamma(1/2+\mu)e^{f(\mu)}k^{+}(\mu,\theta_o)k^{+}(\mu,\chi_o)\right\} \times \left\{\sqrt{2}P_{-1/2}(\cos\chi_o)\Gamma(1/2-\mu)e^{-f(\mu)}k^{-}(\mu,\theta_o)k^{-}(\mu,\chi_o)\right\}$$

where the $f(\mu)$ will be so determined as to render the braced terms algebraic in their behavior at infinity. Let us call the first braced term $K^{+}(\mu,\theta_o)$ and the second $K^{-}(\mu,\theta_o)$. From equation (5.15) $K^{+}(\mu,\theta_o)$ is asymptotically, in the region $|\arg\mu| \leq \pi/2$,

$$K^{+}(\mu,\theta_o) \sim \sqrt{2}P_{-1/2}(\cos\theta_o)A(\theta_o)A(\chi_o)\left\{\frac{\Gamma(1/2+\mu)}{\Gamma(\frac{\theta_o\mu}{\pi}+3/4)\Gamma(\frac{\chi_o}{\pi}\mu+3/4)}\right\} \times$$
$$\Gamma^{2}(3/4)\exp\left\{\mu\left[(\frac{\theta_o}{\pi}+\frac{\chi_o}{\pi})\psi(3/4)+\frac{\theta_o}{\pi}B(\theta_o)+\frac{\chi_o}{\pi}B(\chi_o)\right]+f(\mu)\right\},$$

where the factor in braces, which we shall call F, has yet to be expanded asymptotically.

Now this factor, by Stirling's formula*, is asymptotically

$$F \sim \left[\sqrt{\frac{\pi}{2(\theta_o\chi_o)^{1/2}}}\; e^{-\mu\left(\frac{\theta_o}{\pi}\log\frac{\theta_o}{\pi}+\frac{\chi_o}{\pi}\log\frac{\chi_o}{\pi}\right)}\right]\mu^{-1/2} \quad .$$

It follows that, asymptotically, in the region $|\arg\mu| \leq \pi/2$

$$K^+(\mu,\theta_o) \sim \Gamma(3/4)\cdot\frac{1}{\sqrt{2(\theta_o\chi_o)^{1/2}}}\cdot A(\chi_o)A(\theta_o)\sqrt{2}P_{-1/2}(\cos\theta_o)\,\times$$

$$\times\;\exp\left\{\mu\left[\left(\frac{\theta_o}{\pi}+\frac{\chi_o}{\pi}\right)\psi(3/4)+\frac{\theta_o}{\pi}B(\theta_o)+\frac{\chi_o}{\pi}B(\chi_o)-\frac{\theta_o}{\pi}\log\frac{\theta_o}{\pi}-\frac{\chi_o}{\pi}\log\frac{\chi_o}{\pi}\right]+f(\mu)\right\}$$

Now the square bracket is a constant which we designate by t. Then setting $f(\mu) = -t\mu$, we have,

$$K^+(\mu,\theta_o) \sim \left\{\frac{\Gamma^2(3/4)\;P_{-1/2}(\cos\theta_o)}{(\theta_o\chi_o)^{1/4}}\,A(\theta_o)\,A(\chi_o)\right\}\mu^{-1/2}, \quad \mathrm{Re}\,\mu \geq 0. \tag{5.18}$$

Since now, $K^+(-\mu,\theta_o) = K^-(\mu,\chi_o)$, (see equation 5.17 and beginning of section 5.2),

$$K^-(\mu,\theta_o) \sim \left\{\frac{\Gamma^2(3/4)\;P_{-1/2}(\cos\chi_o)}{(\theta_o\chi_o)^{1/4}}\,A(\theta_o)\,A(\chi_o)\right\}(-\mu)^{-1/2}, \quad \mathrm{Re}\,\mu \leq 0 \quad . \tag{5.19}$$

We have thus succeeded in expressing $K(\mu,\theta_o)$ in the form

$$K(\mu) = \sin\theta_o\,K^+(\mu,\theta_o)\,K^-(\mu,\theta_o) \tag{5.20}$$

where $K^+(\mu,\theta_o)$, $K^-(\mu,\theta_o)$ enjoy the properties mentioned on page 18.

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* $\Gamma(\nu) \sim \dfrac{\nu^{\nu-1/2}e^{-\nu}}{\sqrt{2\pi}}$ for $|\arg\nu| < \pi$

6. The Calculation of $A(\nu)$, $B(\nu)$ and $\phi(\nu)$.

From equation (4.2) on setting $\mu = \nu - 1/2$ we get

$$(6.1)\qquad \frac{2\pi\sin\theta_o}{\sin\pi(\mu+1/2)}\, P_{-1/2+\mu}(\cos\theta_0)\, P_{-1/2+\mu}(\cos\chi_o)\phi_+(\mu+1/2) - \frac{u_o}{\mu+\frac{1}{2}} = g_-(\mu+\tfrac{1}{2}) \quad ,$$

which in conjunction with the equation (4.3) and (4.7) yields,

$$(6.2)\qquad \sin\theta_o\, K^+(\mu,\theta_o)K^-(\mu,\theta_o)\Phi_+(\mu) - \frac{u_o}{\mu+\frac{1}{2}} = G_-(\mu)\,.$$

Here we have set

$$\phi_+(\mu+\tfrac{1}{2}) \equiv \Phi_+(\mu)$$

and

$$g_-(\mu+\tfrac{1}{2}) \equiv G_-(\mu) \quad .$$

Since $\phi_+(\mu+\frac{1}{2})$ is assumed to be analytic for $\mathrm{Re}\left(\mu+\frac{1}{2}\right) \geq \delta(\theta_o)$,

and $g_-(\mu+\frac{1}{2})$ is assumed to be analytic for $\mathrm{Re}\left(\mu+\frac{1}{2}\right) \leq \delta(\theta_o)$, [cf. p.16], it follows that $\Phi_+(\mu)$ is analytic in μ for $\mathrm{Re}\,\mu \geq \delta(\theta_o) - \frac{1}{2}$ and $G_-(\mu)$ is analytic in μ for $\mathrm{Re}\,\mu \leq \delta(\theta_o) - \frac{1}{2}$.

Dividing (6.2) through by $K^-(\mu,\theta_o)$, we get

$$\sin\theta_o\, K^+(\mu,\theta_o)\,\Phi_+(\mu) - \frac{u_o}{(\mu+\frac{1}{2})K^-(\mu,\theta_o)} = G_-(\mu)/K^-(\mu,\theta_o) \quad ,$$

or, rearranging,

$$(6.3)\ \sin\theta_o K^+(\mu,\theta_o)\Phi_+(\mu) - \frac{u_o}{(\mu+1/2)K^-(-1/2,\theta_o)} = \frac{G_-(\mu)}{K^-(\mu,\theta_o)} - \frac{u_o}{(\mu+\frac{1}{2})}\cdot\left[\frac{1}{K^-(-\frac{1}{2},\theta_o)} - \frac{1}{K^-(\mu,\theta_o)}\right]$$

It will be noticed that (6.3) has been so written that the left hand side is analytic for $\mathrm{Re}\mu \geq \delta(\theta_o) - 1/2$ for any $\delta(\theta_o) > 0$ whereas the right hand side is analytic for $\mathrm{Re}\mu \leq \delta(\theta_o) - 1/2$ provided $\delta(\theta_o)$ is so chosen that $K^-(\mu,\theta_o)$ has no

zeros in $\text{Re}\mu \leq \delta(\theta_o)-\frac{1}{2}$. Now $K^-(\mu,\theta_o) = 2P_{-1/2}(\cos\chi_o)\Gamma(\frac{1}{2}-\mu)\cdot e^{-t\mu}k^-(\mu,\theta_o)k^-(\mu,\chi_o)$ so that $K^-(\mu,\theta_o)$ has only positive zeros, i.e., those of $k^-(\mu,\theta_o)$ and $k^-(\mu,\chi_o)$. The first positive zeros are $\mu_1(\theta_o)$ and $\mu_1(\chi_o)$(cf. equation (5.6)). These zeros vary in terms of θ_o or χ_o, in the following manner.

$\cos\theta_o$	$\mu_1(\theta_o)$	$\mu_1(\pi-\theta_o)$
1.0	$\gg 15.0$	1.0
0.6	2.5	1.2
0.2	2.2	1.3
0.0	1.5	1.5
$\cos(\pi-\theta_o)$	$\mu_1(\pi-\theta_o)$	$\mu_1(\theta_o)$

{The values of $\mu_1(\theta_o)$ and $\mu_1(\pi-\theta_o)$ were read from a graph [cf.(4) p.108]}

The smallest possible value of μ_1 is therefore 1.
Since $0 < \theta_o < \pi$, the choice $\delta(\theta_o) \leq \delta < 1$ ensures that, for all θ_o, $K^-(\mu,\theta_o)$ will be free of zeroes and poles when $\text{Re}\,\mu < \frac{1}{2}$. This is to be inferred from the above table, and from the properties of the gamma function involved in $K^-(\mu,\theta_o)$.

Now for any δ, $0 < \delta < 1$, the right hand side of (6.3) is the analytic continuation of the left hand side into the region $\text{Re}(\mu+1/2) \leq \delta$ and the left hand side continues the right hand side analytically into $\text{Re}(\mu+1/2) \geq \delta$. An analytic function is thus defined by (6.3).

If now, $\Phi(\mu)\sim\mu^{-q}$, $q > -1/2$, as $|\mu|\to\infty$ in the right half plane, then, since $K^+(\mu,\theta_o)\sim\mu^{-1/2}$, we can assert that the left hand side of (6.3) vanishes at infinity. Finally if we assume $G_-(\mu)\sim\mu^{-p}$ as $|\mu|\to\infty$ in the left half plane, with $p > 1/2$, then since $K^-(\mu,\theta_o)\sim(-\mu)^{-1/2}$ as $|\mu|\to\infty$, we see that the right hand side vanishes at infinity. Invoking Liouville's theorem we have that the entire function defined by either side of equation (6.3) is identically zero. Hence

$$(6.4) \qquad \Phi_+(\mu) = \frac{u_o}{\sin\theta_o K^-(-\frac{1}{2};\theta_o)K^+(\mu,\theta_o)(\mu+\frac{1}{2})} \quad {}^{**}$$

** Since $K^+(\mu,\theta_o)\sim\mu^{-1/2}$ in the region $\text{Re}\mu \geq 0$ then, clearly, $\Phi_+(\mu)\sim\mu^{-1/2}$ in this region. Also note that $G_-(\mu)$ is given explicitly by

$$G_-(\mu) = \frac{u_o}{\mu+\frac{1}{2}}\left[\frac{K^-(\mu,\theta_o)}{K^-(-\frac{1}{2},\theta_o)} - 1\right] ,$$

(continued on next page)

Note, since $(K^+(\mu,\theta_0))^{-1}$ is zeroless for $\mathrm{Re}\mu > -\frac{1}{2}$, that $\Phi_+(\mu)$ is analytic for $\mathrm{Re}\mu > -\frac{1}{2}$. Substituting $\Phi_+(\mu)$ into (3.20) page 14 (setting $\mu = \nu - 1/2$) and solving for $A(\nu)$ we get

$$(6.5)\qquad A(\nu) = \frac{2\pi u_0}{K^-(-1/2,\theta_0)} \cdot \frac{P_{\nu-1}(\cos\chi_0)}{\nu\sin\pi\nu K^+(\nu-1/2,\theta_0)} .$$

Similarly,

$$(6.6)\qquad B(\nu) = \frac{2\pi u_0}{K^-(-1/2,\theta_0)} \cdot \frac{P_{\nu-1}(\cos\theta_0)}{\nu\sin\pi\nu K^+(\nu-1/2,\theta_0)}$$

Now

$$P_{\nu-1}(\cos\chi_0) = P_{-1/2+\mu}(\cos\chi_0) = P_{-1/2}(\cos\chi_0)k^+(\mu,\chi_0)k^-(\mu,\chi_0)$$

$$P_{\nu-1}(\cos\theta_0) = P_{-1/2+\mu}(\cos\theta_0) = P_{-1/2}(\cos\theta_0)k^+(\mu,\theta_0)k^-(\mu,\theta_0)$$

and

$$\sin\pi(\mu+1/2) = \cos\pi\mu = \frac{\pi}{\Gamma(1/2+\mu)\Gamma(1/2-\mu)} ,$$

so that from (6.5) above and equation (4.9) page 18

$$A(\mu+1/2) = \frac{2u_0}{\sqrt{2}\,P_{-1/2}(\cos\theta_0)K^-(-1/2,\theta_0)} \quad \frac{\Gamma(1/2+\mu)\Gamma(1/2-\mu)P_{-1/2}(\cos\chi_0)k^-(\mu,\chi_0)k^+(\mu_0\theta_0)}{(\mu+1/2)\Gamma(1/2+\mu)e^{t\mu}k^+(\mu,\theta_0)k^+(\mu\,\chi_0)}$$

$$= \frac{2u_0}{(\sqrt{2})^2P_{-1/2}(\cos\theta_0)K^-(-1/2,\theta_0)} \cdot \frac{\sqrt{2}\,P_{-1/2}(\cos\chi_0)\Gamma(1/2-\mu)e^{-t\mu}\,k^-(\mu,\chi_0)k^-(\mu,\theta_0)}{\Gamma\ (\mu+1/2)k^+(\mu,\theta_0)k^-(\mu,\theta_0)}$$

$$= \frac{u_0}{K^-(-1/2,\theta_0)} \cdot \frac{K^-(\mu,\theta_0)}{(\mu+1/2)P_{-1/2+\mu}(\cos\theta_0)} .$$

Or, on setting $\nu = \mu + 1/2$,

$$(6.7)\qquad A(\nu) = \frac{u_0}{K^-(-1/2,\theta_0)} \cdot \frac{K^-(\nu-1/2,\theta_0)}{\nu P_{\nu-1}(\cos\theta_0)} .$$

From equation (3.9) we have, therefore,

$$(6.8)\qquad B(\nu) = \frac{u_0}{K^-(-1/2,\theta_0)} \cdot \frac{K^-(\nu-1/2,\theta_0)}{\nu P_{\nu-1}(\cos\chi_0)} .$$

** (continued from previous footnote)

so that in the region $\mathrm{Re}\mu \leq 0$, $G_-(\mu) \sim \mu^{-1}$. It follows that $\Phi_+(\mu)$, $G_-(\mu)$ do indeed behave properly for large $|\mu|$ in their respective regions. (See the above arguments and the conditions of $\phi_+(\mu+\frac{1}{2})$ on pages 15 and 16).

As a check on these results let us see what happens at $\theta_o = \pi/2 = \chi_o$. Now for $\mu = \nu - 1/2$, from page 18, we have

$$K^-(\nu-\tfrac{1}{2},\theta_o) = \sqrt{2}\, P_{-1/2}(\cos\chi_o) e^{-t\mu}\, \Gamma(1/2-\mu) k^-(\mu,\theta_o) k^-(\mu,\pi-\theta_o)\ ,$$

where $t = -\left[\left(\frac{\theta_o}{\pi} + \frac{\chi_o}{\pi}\right)\psi(3/4) + \frac{\theta_o}{\pi} B(\theta_o) + \frac{\chi_o}{\pi} B(\theta_o) - \frac{\theta_o}{\pi}\log\frac{\theta_o}{\pi} - \frac{\chi_o}{\pi}\log\frac{\chi_o}{\pi}\right]$

(see page 23), and where

$$\frac{k^-(\mu,\theta_o)\Gamma\left(-\frac{\theta_o\mu}{\pi} + 3/4\right)}{\Gamma(3/4)e^{\frac{\theta_o\psi(3/4)\mu}{\pi}}} = \frac{\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_m(\theta_o)}\right)e^{-\frac{\mu}{\mu_m(\theta_o)}}}{\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}\right)e^{-\frac{\mu}{\frac{\pi}{\theta_o}(m-1/4)}}}$$

(see page 21,(5.12)). At $\theta_o = \pi/2$ we have

$$t = -\left[\psi\left(\tfrac{3}{4}\right) + \log 2\right] \tag{6.9}$$

$$k^-(\mu,\pi/2) = \frac{\Gamma(3/4)e^{\frac{\psi(3/4)\mu}{2}}}{\Gamma(-\mu/2+3/4)} \tag{6.10}$$

so that

$$K^-(\mu,\pi/2) = \sqrt{2}\, P_{-1/2}(0)\, \Gamma^2\left(\tfrac{3}{4}\right)\frac{\Gamma(1/2-\mu)}{\Gamma^2(3/4-\mu/2)}\cdot 2^{\mu} \tag{6.11}$$

Setting $\mu = \nu - 1/2$

$$K^-(\nu-1/2,\pi/2) = \frac{2^{\nu}\,\Gamma^2\left(\frac{3}{4}\right)\Gamma(1-\nu)\, P_{-1/2}(0)}{\Gamma^2(1-\nu/2)}$$

Now, from page 19, equation (5.5),

$$P_{-1/2}(0) = \sqrt{\pi}\Big/\Gamma^2(3/4)$$

so that

(continued from previous footnote)

Note also that $G_-(\mu)$ is analytic in μ for $\mathrm{Re}\mu < 1/2$ since the first pole at $K^-(\mu,\theta_o)$ occurs at $\mu = 1/2$. It follows $g_-(\mu+1/2)$ is analytic in $(\mu+1/2)$ for $\mathrm{Re}(\mu+1/2) = \mathrm{Re}\nu \le \delta < 1$. Since now $\Phi_+(\mu)$ is analytic for $\mathrm{Re}\mu > 1/2$, then $\phi_+(\mu+1/2)$ is analytic in $(\mu+1/2)$ for $\mathrm{Re}(\mu+1/2) > 0$. Thus $\phi_+(\nu)$ and $g_-(\nu)$ have a region of analyticity, $0 < \mathrm{Re}\nu \le \delta < 1$, in common, where $\nu = \mu+1/2$ as before.

(6.12) $$K^-(\nu-1/2,\pi/2) = \sqrt{\pi}\; 2^{\nu} \frac{\Gamma(1-\nu)}{\Gamma^2(1-\nu/2)}$$

and hence at $\nu = 0$

(6.13) $$K^-(-1/2,\pi/2) = \sqrt{\pi} = K^+(1/2,\pi/2).$$

Since now

(6.14) $$\Gamma(2z) = \frac{1}{2\sqrt{\pi}}\; 2^{2z}\Gamma(z)\Gamma(z+1/2) \quad ; \quad (\text{cf. } [2], \text{ p.1.}),$$

we have that

$$\Gamma\left[2\left(\tfrac{1-\nu}{2}\right)\right] = \frac{1}{\sqrt{\pi}}\; 2^{-\nu}\,\Gamma\left(\tfrac{1-\nu}{2}\right)\Gamma\left(1-\tfrac{\nu}{2}\right).$$

It follows therefore, that

(6.15) $$K^-(\nu-1/2,\pi/2) = \frac{\Gamma\left(\frac{1-\nu}{2}\right)}{\Gamma(1-\nu/2)} \quad .$$

Since, when $\theta_o = \pi/2$, $K^+(\nu-1/2,\pi/2) = K^-(1/2-\nu,\pi/2)$, then we have also

(6.16) $$K^+(\nu-1/2,\pi/2) = \frac{\Gamma(\nu/2)}{\Gamma\left(\frac{1+\nu}{2}\right)} \quad .$$

It follows from (6.12) and (6.16) that the K's individually reduce to those in the case of the disc. [cf. p. 5, equation (1.18)]

Finally from (6.7) at $\theta_o = \pi/2$ we have, in virtue of (6.15) and (5.5)

$$A(\nu) = \frac{u_o}{\pi}\; \frac{\Gamma\left(\frac{1-\nu}{2}\right)\Gamma\left(\frac{1+\nu}{2}\right)}{\nu} \quad .$$

Thus $A(\nu)$ for $\theta_o = \pi/2$ checks with the $A(\nu)$ of page 7, equation (1.21).

7. Series Expansion of the Potential Integrals

We are now in a position to give explicit power series expansions for $u(r,\theta_0)$ in the regions indicated in Figure 5

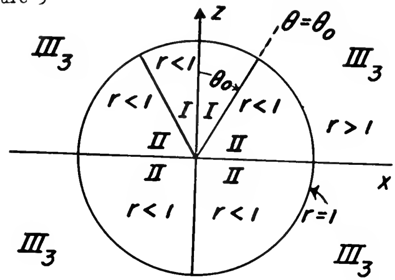

Figure 5.

7.1 In Region I ($0 \le \theta \le \theta_0$, $0 < r < 1$).

We have denoted by C_δ the line $\delta + it$, with $-\infty < t < \infty$. Let C^ρ_δ be that segment of C_δ with t confined to $-\rho \le t \le \rho$, and C^ρ_- be the contour shown in Figure 6 below.

We shall assume in the following that, as $\rho \to \infty$, the potential integral over C^ρ_- vanishes**.

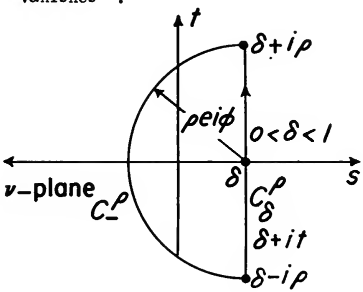

Figure 6.

Now from equations (3.6) and (6.7)

$$(7.1)\quad u(r,\theta) = \frac{1}{2\pi i}\,\frac{u_0}{K^-(-\frac{1}{2},\theta_0)}\int_{C_\delta}\frac{r^{-\nu}K^-(\nu-1/2,\theta_0)P_{\nu-1}(\cos\theta)}{\nu P_{\nu-1}(\cos\theta_0)}\,d\nu$$

In the region surrounded by C^ρ_δ and C^ρ_-, the only poles of the integrand are at $\nu = 0$ and $\nu = -\nu_m(\theta_0)$, where $-\nu_m(\theta_0)$ are the negative zeros of $P_{\nu-1}(\cos\theta_0)$*. Note that

$$K^-(\nu-\tfrac{1}{2},\theta_0) = \sqrt{2}\,P_{-1/2}(\cos\chi_0)e^{-t(\nu-1/2)}\,\Gamma(1-\nu)k^-(\nu-1/2,\theta_0)k^-(\nu-1/2,\chi_0),$$

so that for $0 < \nu < 1$ the Γ function factor has no poles in this region.

** See Note 3, assertion(a-2)

* Since $P_{\nu-1}(\cos\theta_0) = P_{-\nu}(\cos\theta_0)$ it follows $\nu_m(\theta_0)$'s are the positive roots of $P_\nu(\cos\theta_0)$.

Evaluating the integral in (7.1) by residues we have for $0 < r < 1$ and $0 \leq \theta \leqslant \theta_o$,

$$(7.2)\quad u(r,\theta) = u_o - \frac{u_o}{K^-(-\frac{1}{2},\theta_o)} \sum_{m=1}^{\infty} \frac{r^{\nu_m(\theta_o)} K^-(-\nu_m(\theta_o)-\frac{1}{2},\ \theta_o)}{\nu_m(\theta_o)} \left.\frac{P_{\nu_m(\theta_o)}(\cos\theta)}{\frac{\partial}{\partial\nu}P_{\nu-1}(\cos\theta_o)}\right|_{\nu = -\nu_m(\theta_o)}$$

When $\theta_o = \pi/2$ the $\nu_m(\theta_o)$'s are the positive zeros of $P_\nu(\cos \pi/2)$ that is, the positive zeros of $\sqrt{\pi} \,/\, \Gamma(1+^{\nu}/2)\,\Gamma(\frac{1-\nu}{2})$ (see page 19). Thus

$$\nu_m(\theta_o) = 2m + 1 \qquad\qquad m = 0,\ 1,\ 2, \cdots$$

From page 28, equations (6.13) and (6.15) respectively, we have,

$$K^-(-1/2,\pi/2) = \sqrt{\pi} \text{ and } K^-(\nu-1/2,\pi/2) = \frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(1-\nu/2)}$$

Therefore

$$\frac{r^{2m+1}}{2m+1} K^-(-2m-3/2,\pi/2) \left.\frac{P_{2m+1}(\cos\theta)}{\frac{\partial}{\partial\nu} P_{\nu-1}(\cos \pi/2)}\right|_{\nu= -(2m+1)}$$

$$= \lim_{\nu \to -(2m+1)} \left\{ (\nu+2m+1 \left[\left(\frac{\Gamma(\frac{1-\nu}{2})}{\cancel{\Gamma(\frac{1-\nu}{2})}}\right)\left(\frac{\Gamma(\frac{1}{2}+\frac{\nu}{2})\,\cancel{\Gamma(\frac{1-\nu}{2})}}{\sqrt{\pi}}\right)\right]\right\} P_{2m+1}(\cos\theta) \cdot \frac{r^{2m+1}}{2m+1}$$

$$= \lim_{\nu \to -(2m+1)} \left\{ (\nu+2m+1)\,\frac{\sqrt{\pi}}{\cos\frac{\pi\nu}{2}} \right\} \frac{r^{2m+1}}{2m+1} \cdot P_{2m+1}(\cos\theta)$$

$$= \frac{\sqrt{\pi}}{2m+1}\,\frac{P_{2m+1}(\cos\theta)}{(\frac{\pi}{2})\sin\pi\frac{2m+1}{2}} = \frac{2}{\sqrt{\pi}}\,(-1)^m \cdot \frac{r^{2m+1}}{2m+1} \cdot P_{2m+1}(\cos\theta)$$

We have therefore at $\theta_o = \pi/2$

$$(7.3)\quad u(r,\theta) = u_o - \frac{2u_o}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{r^{2m+1}}{2m+1}\ P_{2m+1}(\cos\theta)$$

7.2 In Region II

For Region II, [$(0 < \; < 1,\ \theta_o \leqslant \theta \leq \pi)$, see figure 5], we have an expression similar to (7.2), the only difference being that $\nu_m(\chi_o)$ is substituted for $\nu_m(\theta_o)$, and χ_o for θ_o, in $\frac{\partial}{\partial\nu} P_{\nu-1}(\cos\theta_o)$. This fact follows immediately from the form of the expression for $B(\nu)$ in equation (6.8). Thus in region II, the expression for $u(r,\theta)$ is,

(7.4)
$$u(r,\theta) = u_o - \frac{u_o}{K^-(-\frac{1}{2},\theta_o)} \sum_{m=1}^{\infty} r^{\nu_m(\chi_o)} \cdot \left(\frac{K^-(-\nu_m(\chi_o)-\frac{1}{2},\theta_o)}{\nu_m(\chi_o)}\right) \frac{P_{\nu_m(\chi_o)}(-\cos\theta)}{\frac{\partial}{\partial\nu} P_{\nu-1}(\cos\chi_o)}\Bigg|_{\nu=-\nu_m(\chi_o)}$$
$$\theta_o \leq \theta \leq \pi,\ r < 1.$$

7.3 Region III

To arrive at an eigenfunction expansion for $u(r,\theta)$ in region III $(1 < r < \infty,\ 0 \leq \theta \leq \pi)$, we use the expression for $A(\nu)$ given in equation (6.5) instead of the one given in (6.7). Thus we take

$$A(\nu) = \frac{2\pi u_o}{K^-(-\frac{1}{2},\ \theta_o)} \cdot \frac{P_{\nu-1}(\cos\chi_o)}{\nu\sin\pi\nu\ K^+(\nu-\frac{1}{2},\ \theta_o)} \quad .$$

Employing this $A(\nu)$, the integral expression for $u(r,\theta)$ becomes,

$$(7.5)\quad u(r,\theta) = \frac{1}{2\pi i} \cdot \frac{2\pi u_o}{K^-(-1/2,\theta_o)} \int_{C_\delta} \frac{r^{-\nu} P_{\nu-1}(\cos\chi_o) P_{\nu-1}(\cos\theta_o)\, d\nu}{\nu\sin\pi\nu\ K^+(\nu-1/2,\theta_o)}$$

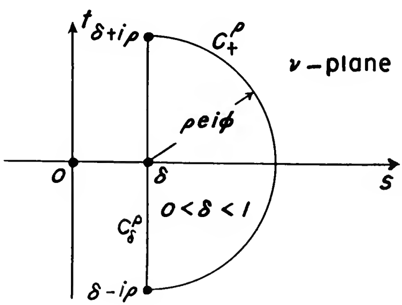

Figure 7

Remembering $0 < \delta < 1$, then in the region bounded by C_δ^ρ and C_+^ρ (Figure 7), the only poles in the integrand occur at $\nu = p$ for $p = 1,2,\ldots$. It follows, under the assumption that the contribution of C_+^ρ vanishes as $p \to \infty$ *, that in the région $0 \le \theta \le \pi$, $r > 1$,

$$u(r,\theta) = \frac{-2u_o}{K^-(-\frac{1}{2}, \theta_o)} \sum_{p=0}^{\infty} r^{-(2p+1)} \cdot \frac{P_{2p}(\cos \chi_o) P_{2p}(\cos \theta)}{(2p+1)K^+(2p + \frac{1}{2}, \theta_o)}$$

(7.6)
$$+ \frac{2u_o}{K^-(-\frac{1}{2}, \theta_o)} \sum_{p=1} r^{-2p} \cdot \frac{P_{2p-1}(\cos \chi_o) P_{2p-1}(\cos \theta)}{2pK^+(2p + \frac{1}{2}, \theta_o)} ,$$

where we have separated the odd and even powers of r.

Let us inspect this expression at $\theta_o = \frac{\pi}{2}$. First of all we know that

$$P_\nu(0) = \frac{\sqrt{\pi}}{\Gamma(\frac{\nu}{2} + 1)\Gamma(\frac{1-\nu}{2})} . \qquad (7.7)$$

(Page 19, equation (5.5))

It follows $P_{2p-1}(0) = 0$ for $p = 1,2,\ldots$, so that in (7.6) the even powers of r vanish.

Defining $M(p,\theta_o)$ to be

$$M(p,\theta_o) = \frac{P_{2p}(\cos \theta_o)}{K^+(2p + \frac{1}{2}, \theta_o)}$$

then at $\frac{\pi}{2}$, $M(p,\pi/2)$ reduces to

$$M(p,\pi/2) = \frac{P_{2p}(0)}{K^+(2p + \frac{1}{2}, \pi/2)} .$$

Now from (6.16) we have

$$K^+(2p + \frac{1}{2}, \pi/2) = \frac{\Gamma(p + 1/2)}{\Gamma(p + 1)}$$

and from (7.8) we know that

$$P_{2p}(0) = \frac{\sqrt{\pi}}{\Gamma(p+1)\Gamma(\frac{1}{2} - p)} .$$

---

* See Note 3, assertion(a - 2).

Hence,

$$M(p,\pi/2) = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2} - p)\Gamma(\frac{1}{2} + p)}$$

But from [2], page 1, we have ,

$$\Gamma(\tfrac{1}{2} - p)\Gamma(\tfrac{1}{2} + p) = \pi/ \cos \pi p \ .$$

Therefore,

$$M(p,\pi/2) = \frac{1}{\sqrt{\pi}} \cos \pi p = \frac{(-1)^p}{\sqrt{\pi}} \qquad p = 0,1,2,3$$

Hence, for $\theta_o = \frac{\pi}{2}$, since $K^+(1/2,\pi/2) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, (equation (6.13)),

$$(7.9)\quad u(r,\theta) = \frac{2u_o}{\pi} \sum_{p=0}^{\infty} (-1)^{p+1} \frac{P_{2p}(\cos \theta)}{2p+1} r^{-2p+1} \qquad r > 1,\ 0 \le \theta \le \pi \ .$$

Equations (7.5) and (7.9), representing the power series expansions of the potential for the case of the disc can be summed. It can be shown* that for $r < 1$ and $r > 1$ the result is

$$(7.10)\quad u(r,\theta) = \frac{2u_o}{\pi} \arctan\left\{\sqrt{\frac{2}{r^2-1 + \sqrt{(r^2-1)^2+4r^2\cos^2\theta}}}\right\} \ .$$

This is the well known form usually obtained via the use of ellipsoidal harmonics. [cf. [5], p. 112.].

8. The Capacitance of the Cone

From equation (7.6) we see that $u(r,\theta)$ can be written as

$$(8.1)\quad u(r,\theta) = \frac{A(\theta)}{r} + \frac{B(\theta)}{r^2} + \ldots .$$

where

$$(8.2)\quad A(\theta) = \frac{-2u_o}{K^-(-1/2,\theta_o)K^+(1/2,\theta_o)} \ .$$

From this relation it follows (see page 9, equation (2.3)), that the capacitance, $C(\theta_o)$, is given by the formula:

$$(8.3)\quad C(\theta_o) = \frac{2}{K^-(-1/2,\theta_o)K^+(1/2,\theta_o)} \ , \qquad 0 < \theta_o < \pi \ .$$

---

* See Note 4

From the definitions of $K^+(\mu,\theta_o)$ and $K^-(\mu,\theta_o)$ [equation (4.9) page 18] we have,

$$K^-(-1/2,\theta_o) = \frac{P_{-1/2}(\cos \chi_o)}{P_{-1/2}(\cos \theta_o)} \quad K^+(1/2,\theta_o) \quad ,$$

where $\chi_o = \pi - \theta_o$.

As a result we may write the capacitance C as

$$(8.4) \quad C(\theta_o) = \frac{2P_{-1/2}(\cos \theta_o)}{P_{-1/2}(\cos \chi_o)(K^+(1/2,\theta_o))^2}$$

At $\theta_o = \pi/2$ - that is when the cone becomes a disc - we know that $K^-(-1/2,\pi/2) = K^+(1/2,\pi/2) = \sqrt{\pi}$.

From (8.3) we have then

$$(8.5) \qquad C(\pi/2) = \frac{2}{\pi} \quad .$$

This is the well known result, [cf. (5) p.112].

9. The Charge Density at Singular Points of the Cone

It is of interest to see how the charge density behaves at the apex and at the circular edge of the cone.

9.1 The Charge Densities at the Apex of the Cone

Employing (7.2) we compute* $-\frac{1}{4\pi r} \left.\frac{\partial u(r,\theta)}{\partial\theta}\right| \theta = \theta_o^-$. The result for $0 < r < 1$ will give us the charge density $\omega(r,\theta_o^-)$ on the inside of the cone.

$$(9.1) \quad \omega(r,\theta_o^-) = \frac{1}{4\pi r} \frac{u_o}{K^-(-\frac{1}{2},\theta_o)} \sum_{m=1}^{\infty} r^{\nu_m(\theta_o)} \frac{K(-\nu_m(\theta_o)-\frac{1}{2},\theta_o)}{\nu_m(\theta_o)} \frac{\left.\frac{\partial}{\partial\theta} P_{\nu_m(\theta_o)}(+\cos\theta)\right|_{\theta=\theta_o}}{\left.\frac{\partial}{\partial\nu} P_{\nu-1}(\cos\theta_o)\right|_{\nu=-\nu_m(\theta_o)}}$$

Employing (7.4) the charge density on the outside surface of the cone $\omega(r,\theta_o^+)$ is*

$$(9.2) \quad \omega(r,\theta_o^+) = \frac{1}{4\pi r} \frac{u_o}{K^-(-\frac{1}{2},\theta_o)} \sum_{m=1}^{\infty} r^{\nu_m(\chi_o)} \frac{K(-\nu_m(\chi_o)-\frac{1}{2},\theta_o)}{\nu_m(\chi_o)} \frac{\left.\frac{\partial}{\partial\theta} P_{\nu_m(\chi_o)}(-\cos\theta)\right|_{\theta=\theta_o}}{\left.\frac{\partial}{\partial\nu} P_{\nu-1}(\cos\theta_o)\right|_{\nu=-\nu_m(\chi_o)}}$$

---

* The termwise differentiation with respect to θ of (7.2)((7.3)) and the setting of $\theta = \theta_o^+(\theta_o^-)$ to arrive at equation (9.1),((9.2)) is justified in Note 3 assertions (c-2) and (c-3).

Using the graph on page 108 of [4], we can find $\nu_1(\theta_o)$ and $\nu_1(\chi_o)$, the first positive roots of $P_\nu(\cos\theta_o)$ and $P_\nu(\cos\chi_o)$ respectively. With this knowledge the behavior at $\omega(r,\theta_o^+)$ and $\omega(r,\theta_o^-)$ as $r \longrightarrow 0$ can be tabulated as follows:

$\cos\theta_o$	$\nu_1(\theta_o)$	$\nu_1(\chi_o)$	$\omega(r,\theta_o^-)$ at $r=0$	$\omega(r,\theta_o^+)$ at $r=0$
1.0	$\gg 15.0$	0.5	$\ll r^{15.0}$	$\sim r^{-0.5}$
0.6	2.1	0.6	$\sim r^{1.1}$	$\sim r^{-0.4}$
0.2	1.2	0.8	$\sim r^{0.2}$	$\sim r^{-0.2}$
0.0	1.0	1.0	$\sim r^{0.0}=1$	$\sim r^{-0.0}=1$
$\cos\chi_o$	$\nu_1(\chi_o)$	$\nu_1(\theta_o)$	$\omega(r,\theta_o^+)$ at $r=0$	$\omega(r,\theta_o^-)$ at $r=0$

9.2 The Charge Densities at the Circular Edge.

We shall derive the behavior of the charge densities at the circular edge from the equations (9.1) and (9.2). To this end we employ the following theorem [cf.(6).pp 181-182];

If

$$(9.3)\qquad f(s) = \int_0^\infty e^{-st}\,d\alpha(t) \qquad\qquad s > 0$$

and if for some number γ

$$(9.4)\qquad \alpha(t) \sim \frac{At^\gamma}{\Gamma(\gamma+1)} \qquad\qquad \text{as } t \longrightarrow \infty$$

where A is a constant, then

$$(9.5)\qquad f(s) \sim A/s^\gamma \qquad\qquad \text{as } s \longrightarrow 0^+$$

The integral in (9.3) is the ordinary Stieltjes integral and it has been assumed that $\alpha(x)$ is normalized, that is

$$(9.6)\qquad \begin{aligned} &\alpha(0) = 0 \\ &\alpha(x) = \frac{\alpha(x^+) + \alpha(x^-)}{2} \qquad \text{at every point of discontinuity.} \end{aligned}$$

Now on setting $t_m = \nu_m(\theta_o) - 1$, equation (9.1) can be written as,

$$(9.7)\quad \omega(r,\theta_o^-) = \frac{u_o}{4\pi K^-(-\frac{1}{2},\theta_o)} \sum_{m=1}^{\infty} r^{t_m} \frac{K^-(-t_m-\frac{3}{4},\theta_o)}{t_m+1} \frac{\frac{\partial}{\partial\theta} P_{t_m+1}(\cos\theta)}{\frac{\partial}{\partial\nu} P_{\nu-1}(\cos\theta)} \Bigg|_{\substack{\theta=\theta_o \\ \nu=-(t_m+1)}}$$

Setting $r = e^{-s}$ for $0<r<1$ and expressing the jump in $\alpha(t)$ in the neighborhood of $t = t_m$ in the form,

$$d\alpha(t)\Big]_{t=t_m} = \frac{K^-(-t_m-3/4,\theta_o)}{t_m+1} \frac{\frac{\partial}{\partial\theta} P_{t_m+1}(\cos\theta)}{\frac{\partial}{\partial\nu} P_{\nu-1}(\cos\theta)} \Bigg|_{\substack{\theta=\theta_o \\ \nu=-(t_m+1)}} ,$$

we can write

$$(9.8)\quad \omega(e^{-s},\theta_o^-) = \frac{u_o}{4\pi K^-(-\frac{1}{2},\theta_o)} \int_0^{\infty} e^{-st} d\alpha(t) \quad .$$

Thus in order to ascertain the behavior of $\omega(r,\theta_o^-)$ in the vicinity of $r=1$, we need only find the asymptotic behavior of $\alpha(t)$. To this end we first find the asymptotic behavior of $d\alpha(t)\big|_{t=t_m}$.

Now from equation (5.18), and the fact that $K^-(-\mu,\theta_o)$ $= \frac{P_{-1/2}(\cos\chi_o)}{P_{-1/2}(\cos\theta_o)} K^+(\mu,\theta_o)$, we know that, for $\nu-1/2 = \mu$,

$$(9.9)\quad K^-(-(\nu-\tfrac{1}{2})\theta_o) \sim \frac{\Gamma(3/4) P_{-1/2}(\cos\chi_o)}{(\theta_o\chi_o)^{1/4}} A(\theta_o)A(\chi_o)(\nu-\tfrac{1}{2})^{-1/2}; \quad \mathrm{Re}(\nu-\tfrac{1}{2}) > 0 .$$

Also, since

$$\frac{\partial}{\partial\theta} P_\nu(\cos\theta_o) = -(\nu)(\nu-1)P_\nu^{-1}(\cos\theta_o) \quad , \quad [\text{cf. } [2] \text{ page } 63],$$

Then, because of the relation (cf. [2] p. 71)

$$(9.10)\quad \nu^{-\mu} P_\nu^{\mu}(\cos\theta_o) = \sqrt{\frac{2}{\pi\nu\sin\theta_o}} \cos\left((\nu+\tfrac{1}{2})\theta_o - \pi/4 + \mu\pi/2\right)\left[1 + O(\nu^{-3/2})\right],$$

(where $\epsilon \le \theta_o \le \pi-\epsilon$, $\epsilon > 0$, $\nu \ggg \frac{1}{\epsilon}$ and $|\arg\nu| < \pi$),

it is easily seen that

$$(9.11)\quad \frac{\partial}{\partial\theta} P_\nu(\cos\theta) = (\nu-1)\sqrt{\frac{2}{\pi\nu\sin\theta_o}} \left\{\sin\left((\nu+\tfrac{1}{2})\theta_o - \pi/4\right)\right\}\left[1 + O(\nu^{-3/2})\right].$$

To arrive at the behavior of $\frac{\partial}{\partial\nu}P_{\nu-1}(\cos\theta_o)\Big|_{\nu=-(t_m+1)}$ for large t_m, first note that

$$(9.12)\quad \frac{\partial}{\partial\nu}P_{\nu-1}(\cos\theta_o)\Big|_{\nu=-(t_m+1)} = \frac{\partial}{\partial\nu}P_{-\nu}(\cos\theta_o)\Big|_{\nu=-(t_m+1)}$$

$$= -\frac{\partial}{\partial\nu}P_\nu(\cos\theta_o)\Big|_{\nu=+(t_m+1)}$$

since $P_{-\nu-1}(\cos\theta) = P_\nu(\cos\theta)$. Now the equation (9.10) gives, at least for $\pi/6 < \theta < 5/6\pi$, a convergent power series in ν, so that termwise differentiation is permissable. As a result,

$$(9.13)\quad \frac{\partial}{\partial\nu}P_{\nu-1}(\cos\theta_o) = -\theta_o\sqrt{\frac{2}{\pi\nu\sin\theta_o}}\ \sin\left((\nu+\tfrac{1}{2})\theta_o - \frac{\pi}{4}\right)\left[1+O(\nu^{-3/2})\right]$$

From (9.9) and (9.11) and (9.13) we have then, as $t_m \longrightarrow \infty$

$$(9.14)\quad d\alpha(t)\Big|_{t=t_m} \sim \frac{1}{2\chi_m}\cdot\frac{\Gamma^2(3/4)P_{-1/2}(\cos\theta_o)A(\theta_o)A(\chi_o)t_m^{-1/2}}{\sqrt{(\theta_o\chi_o)^{1/2}}}\left(\frac{\chi_m}{\theta_o}\right)$$

It can be then shown that

$$(9.15)\quad \alpha(t)\sim\frac{\Gamma^2(3/4)P_{-1/2}(\cos\theta_o)A(\theta_o)A(\chi_o)}{\sqrt{(\theta_o\chi_o)^{1/2}}}\quad t^{1/2}$$

Because of (9.5), equation (9.8) can now be written, in the vicinity of $s = 0$, as

$$(9.16)\quad \omega(e^{-s},\theta_o^-)\sim\frac{u_o\Gamma^2(3/4)P_{-1/2}(\cos\theta_o)A(\theta_o)A(\chi_o)}{2\pi^2K^-(-1/2,\theta_o)\sqrt{(\theta_o\chi_o)^{1/2}}}\cdot\Gamma(3/2)s^{-1/2},$$

or since $r = e^{-s} = 1 - s$

$$(9.17)\quad \omega(r,\theta_o^-)\sim\frac{u_o\Gamma^2(3/4)P_{-1/2}(\cos\theta_o)A(\theta_o)A(\chi_o)}{2\pi^2K^-(-1/2,\theta_o)\sqrt{(\theta_o\chi_o)^{1/2}}}\cdot\frac{\Gamma(3/2)}{\sqrt{1-r}}\quad .$$

In a similar manner it can be shown that

$$(9.18)\quad \omega(r,\theta_o^+) \sim \frac{u_o \Gamma^2(3/4) P_{-1/2}(\cos\psi_o) A(\theta_o) A(\psi_o)}{2\pi^2 K^-(-1/2,\theta_o)\sqrt{(\theta_o \psi_o)^{1/2}}} \; \frac{\Gamma(3/2)}{\sqrt{1-r}} \;.$$

At $\theta_o = \pi/2$ we know that

$P_{-1/2}(0) = \sqrt{\pi} \,/\, \Gamma^2(3/4)$ (page 27)

$K^-(-1/2,\pi/2) = \sqrt{\pi}$ (equation 6.13 page 20)

$A(\pi/2) = 1$ (see footnote page 21)

Thus

$$\omega(r,\pi/2^+) = \omega(r,\pi/2^-) \sim \frac{u_o \Gamma(3/2)}{2\pi^2\sqrt{1-r} \;\cdot\; \sqrt{\pi}/2}$$

Consequently,

$$(9.19)\quad \omega(r,\pi/2^+) = \omega(r.\pi/2^-) \sim \frac{u_o}{2^{3/2}\pi^2\sqrt{1-r}}$$

since $\Gamma(3/2) = 1/2\sqrt{\pi}$, ([2] page 1).

This result is given in [5], p.112.

Note I: An Infinite Product Expansion of the Legendre Function of the First Kind

In order to justify writing $P_{-1/2+\mu}(\cos\theta_o)$ as

a) $$P_{-1/2+\mu}(\cos\theta_o)=P_{-1/2}(\cos\theta_o)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_m(\theta_o)}\right)\exp\left(\frac{-\mu}{\mu_m(\theta_o)}\right)\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_m(\theta_o)}\right)\exp\left(\frac{+\mu}{\mu_m(\theta_o)}\right)$$

it is sufficient to prove*, for the variable μ, and for fixed θ_o, $0 < \theta_o < \pi$, that $P_{-1/2+\mu}(\cos\theta_o)$ is of finite order ρ with $0 \leq \rho \leq 1$.

If $f(\mu)$ is an entire function, then $f(\mu)$ is said to be of finite order if there is a positive number A such that as $|\mu| \longrightarrow \infty$

b) $$f(\mu) = O(e^{|\mu|^A}).$$

The order ρ of $f(\mu)$ is defined as the lower bound of all A for which equation (b) holds. If $0 \leq \rho \leq 1$ and $f(\mu)$ is an even function of μ then the Hadamard Factorization Theorem (Titchmarsh: Theory of Functions, page 250) permits the writing of $f(\mu)$ as an infinite product in the following form

c) $$f(\mu) = f(0)\prod_{m=1}^{\infty}\left(1+\frac{\mu}{\mu_m}\right)\exp\left(\frac{-\mu}{\mu_m}\right)\cdot\prod_{m=1}^{\infty}\left(1-\frac{\mu}{\mu_m}\right)\exp\left(\frac{\mu}{\mu_m}\right)$$

where is it assumed that $f(0) \neq 0$. The μ_m are of course the roots of $f(\mu)$.

Now from [2] - page 71 we know, for $|\mu - 1/2| \ggg 1$ and $|\arg\mu - 1/2| < \pi$, that

d) $$P_{-1/2+\mu}(\cos\theta_o) = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\mu+1/2)}{\Gamma(\mu+1)}\frac{\cos[\mu\theta_o - \pi/4]}{\sqrt{2\sin\theta_o}}\left[1 + O(1/\mu)\right].$$

Now it is easily verified that the order ρ of $\cos[\mu\theta_o - \pi/4]$ is 1. Furthermore by the Stirling approximation formula, in the region $\arg|\mu - 1/2| < \pi$ the factor in (d) containing the gamma functions, behaves asymptotically like $\mu^{-1/2}$ and hence does not affect the order of $P_{-1/2+\mu}(\cos\theta_o)$. Thus we can say in the region $|\mu-1/2| \ggg 1$ and $|\arg\mu - 1/2| < \pi$, $P_{-1/2+\mu}(\cos\theta_o) = O(e^{|\mu|})$ for $\arg|\mu-1/2| \leq \pi$ and $|\mu-1/2| \ggg 1$. Since $P_{-1/2+\mu}(\cos\theta_o)$ is even in μ it is of order 1 and we may apply equation (c) to arrive at (a).

---

*It should be recalled that $P_{-1/2+\mu}(\cos\theta_o)$ is for fixed θ_o an entire even function of μ and that its zeros are given asymptotically by equation (5.6).

Note 2: <u>The Asymptotic Behavior of a Certain Infinite Product</u>

In this note we shall prove that

a) $$R(\mu) = \prod_{m=1}^{\infty}\left(\frac{1+\mu/\mu_m(\theta_o)}{1+\mu\Big/\frac{\pi}{\theta_o}(m-1/4)}\right)$$

for $|\arg\mu| \leq \pi/2$ approaches the constant $A(\theta_o)$, as $|\mu| \longrightarrow \infty$, where

b) $$A(\theta_o) = \prod_{m=1}^{\infty}\left\{\frac{(m-1/4)}{\frac{\theta_o}{\pi}\mu_m(\theta_o)}\right\} .$$

Let us make the substitution

c) $$\zeta = \frac{1}{\mu} ,$$

and examine the behavior of $R(\frac{1}{\zeta})$ in the neighborhood of $\zeta = 0$ for $|\arg\zeta| \leq \pi/2$.

Now $R(\frac{1}{\zeta})$ can be written as

d) $$R(\tfrac{1}{\zeta}) = \left\{\prod_{m=1}^{\infty}\frac{\frac{\pi}{\theta_o}(m-1/4)}{\mu_m(\theta_o)}\right\}\left\{\frac{\zeta\mu_m(\theta_o)+1}{\zeta(m-1/4)\frac{\pi}{\theta_o}+1}\right\} .$$

From the equation (5.8) on page 19,

e) $$\mu_m(\theta_o) \sim \frac{\pi}{\theta_o}(m-1/4) + \frac{C(\theta_o)}{m} , \qquad 0 < \theta_o < \pi. \; *$$

It follows then that the infinite product

$$\prod_{m=1}^{\infty}\frac{\frac{\pi}{\theta_o}(m-1/4)}{\mu_m(\theta_o)} = A(\theta_o)$$

converges. Furthermore, calling the infinite product of the second factors in d), $N(\zeta)$, that is

f) $$N(\zeta) = \prod_{m=1}^{\infty}\left\{\frac{\zeta\mu_m(\theta_o)+1}{\zeta(m-1/4)\frac{\pi}{\theta_o}+1}\right\} ,$$

then it can be shown that $N(\zeta)$ converges uniformly in all ζ in the region $|\arg\zeta| \leq \pi/2$.

For, $N(\zeta)$ can be written as

g) $$N(\zeta) = \prod_{m=1}^{\infty}\left\{1 + \frac{\zeta\left[-(\frac{\pi}{\theta_o}(m-1/4)+\mu_m(\theta_o)\right]}{\zeta(m-1/4)\frac{\pi}{\theta_o}+1}\right\}$$

If we define $f_m(\zeta)$ as

---

*Here and on page 19, $C(\theta_o)$ and $A(\theta_o)$ must not be confused with the capacitance or related quantities.

h) $$f_m(\zeta) = \frac{-\zeta\left[\frac{\pi}{\theta_o}(m-1/4) - \mu_m(\theta_o)\right]}{\zeta(m-1/4)\,\frac{\pi}{\theta_o} + 1}$$

then $N(\zeta)$ converges uniformly in the region $|\arg\zeta| \leq \pi/2$ if and only if the series

i) $$\sum_{m=1}^{\infty} |f_m(\zeta)|$$

converges uniformly in this region.

Now for large m, because of equation (e),

j) $$|f_m(\zeta)| \leq \frac{|\zeta| \cdot \frac{|C(\theta_o)|}{m}}{|\zeta(m-1/4)\pi/\theta_o + 1|}$$

Since however

k) $$|\zeta(m-1/4)\pi/\theta_o + 1| \geq |\zeta|(m-1/4)\pi/\theta_o \quad \text{for } |\arg\zeta| \leq \pi/2,$$

we can conclude,

l) $$|f_m(\zeta)| \leq \frac{\theta_o C(\theta_o)}{\pi m(m-1/4)} .$$

Thus the series (i) converges uniformly for all ζ, $|\arg\zeta| \leq \pi/2$, and hence so does the right hand side of (g).

We can now write $R(\frac{1}{\zeta})$ as

m) $$R(\tfrac{1}{\zeta}) = \prod_{m=1}^{\infty}\left(\frac{\frac{\pi}{\theta_o}(m-1/4)}{\mu_m(\theta_o)}\right) \cdot \quad N(\zeta)$$

or $$R(\tfrac{1}{\zeta}) = A(\theta_o)\, N(\zeta) .$$

Since $N(\zeta)$ is continuous for all ζ, $|\arg\zeta| \leq \pi/2$, then

$$\lim_{|\zeta| \to 0} R(\tfrac{1}{\zeta}) = A(\theta_o) \cdot \quad N = A(\theta_o) ; \ |\arg\zeta| \leq \pi/2 ,$$

or

p) $$\lim_{|\mu| \to 0} R(\mu) = A(\theta_o) \quad , \quad |\arg\mu| \leq \pi/2 .$$ q. e. d.

Note 3

On the Convergence of the Integral Representations of the Potential Solution $u(r,\theta)$ and its Various Derivatives, and the Verification of the Solution.

In equations (3.6) and (3.7) the potential function for the cone problem, $u(r,\theta)$ was represented as

$$(1)\quad u(r,\theta) = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu} A(\nu)P_{\nu-1}(\cos\theta)d\nu \qquad \begin{matrix} 0 < r < \infty \\ 0 \le \theta \le \theta_o \end{matrix}$$

$$(2)\quad u(r,\theta) = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu}B(\nu)P_{\nu-1}(\cos\chi)d\nu^* \qquad \begin{matrix} 0 < r < \infty \\ \chi_o \le \chi \le \pi \end{matrix}$$

where C_δ is the contour parallel to the $\mathrm{Im}\,\nu$ axis at $\mathrm{Re}\,\nu = \delta(\theta_o)$. In what follows we take $\delta(\theta_o) = 1/2$** once and for all.

We shall prove the following assertions:

(a-1)-The integral in (1) ((2)) converges uniformly for $r \ge \epsilon > 0$ and $0 \le \theta \le \theta$ $(\chi_o \le \chi \le \pi)$. In particular, therefore

$$u(r,\theta_o) = \frac{1}{2\pi i}\int_{C_\delta} r^{-\nu} \begin{Bmatrix} A(\nu) \\ B(\nu) \end{Bmatrix} \begin{Bmatrix} P_{\nu-1}(\cos\theta_o) \\ P_{\nu-1}(\cos\chi_o) \end{Bmatrix} d\nu$$

(a-2)- For $r \ge \epsilon > 0$ the contribution of the integrand in (1)((2)) when integrated over the circular arc C_+^ρ (C_-^ρ) approaches zero uniformly in ϕ, $-\pi/2 \le \phi \le \pi/2$ $(\pi/2 \le \phi \le 3/2\pi)$ as $\rho \longrightarrow \infty$. (See Figure 1)

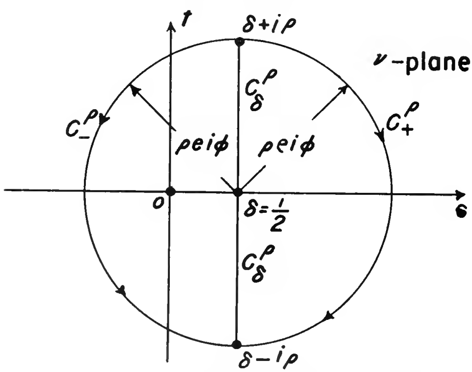

Figure 1.

*As in our previous notation we take $\chi = \pi-\theta$ and $\chi_o = \pi-\theta_o$

** It has been shown on page 25 of the text that any $\delta(\theta_o)$ such that $0 < \delta < 1$, will suffice.

Consequently (1) and (2) may be evaluated by residues. Equation 7.6 then yields $\lim_{r \to \infty} u(r,\theta) \to 0$ uniformly in θ.

(b)—Let us call Region 1 (Region 2) the (r,θ) domain $0 < \epsilon_1 \leq r < \infty$ $(0 \leq \epsilon_2 \leq r < \infty)$ and $0 \leq \theta \leq \theta_0 - \eta_1$ $(\chi_0 + \eta_2 \leq \chi \leq \pi)$.

It can then be shown that the integrals

$$(3)\begin{cases} \displaystyle\int_{C_\delta} A(\nu)\begin{Bmatrix} \dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial r} \end{Bmatrix} r^{-\nu} P_{\nu-1}(\cos\theta)\,d\nu \\ \displaystyle\int_{C_\delta} A(\nu)\begin{Bmatrix} \dfrac{\partial^2}{\partial\theta^2} \\ \dfrac{\partial^2}{\partial r^2} \end{Bmatrix} r^{-\nu} P_{\nu-1}(\cos\theta)\,d\nu \end{cases} \qquad \text{Region 1}$$

$$(4)\begin{cases} \displaystyle\int_{C_\delta} B(\nu)\begin{Bmatrix} \dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial r} \end{Bmatrix} r^{-\nu} P_{\nu-1}(\cos\chi)\,d\nu \\ \displaystyle\int_{C_\delta} B(\nu)\begin{Bmatrix} \dfrac{\partial^2}{\partial\theta^2} \\ \dfrac{\partial^2}{\partial r^2} \end{Bmatrix} r^{-\nu} P_{\nu-1}(\cos\chi)\,d\nu \end{cases} \qquad \text{Region 2}$$

are uniformly convergent for r and θ in regions 1 and 2 respectively. Consequently the operators $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial\theta}$, $\frac{\partial^2}{\partial r^2}$, $\frac{\partial^2}{\partial\theta^2}$, commute with the integral operator in equations (1) and (2) and the result is continuous for all r and θ in the closed regions 1 and 2 respectively. In particular we may say that the integral expressions in (3) and (4) are continuous across $r = 1$, together with their first and second derivatives and the latter obey Laplace's equation.

(c-1) In equations (1) and (2) for $r \geq 1$ $(r \leq 1)$ we may deform the contour C_δ to the contour $C^+_{\pi/4}$ $(C^-_{\pi/4})$ shown in Figure 2 below

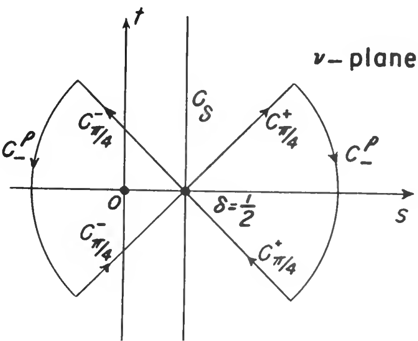

Figure 2.

and

$$(5)\quad u(r,\theta) = \frac{1}{2\pi i}\int_{C^+_{\pi/4}} r^{-\nu}\begin{Bmatrix}A(\nu)\\B(\nu)\end{Bmatrix}P_{\nu-1}\begin{Bmatrix}\cos\theta\\\cos\chi\end{Bmatrix}d\nu, \qquad r \geq 1 \quad \begin{Bmatrix}0 \leq \theta \leq \theta_o\\\chi_o \leq \chi \leq \pi\end{Bmatrix}$$

$$(6)\quad u(r,\theta) = \frac{1}{2\pi i}\int_{C^-_{\pi/4}} r^{-\nu}\begin{Bmatrix}A(\nu)\\B(\nu)\end{Bmatrix}P_{\nu-1}\begin{Bmatrix}\cos\theta\\\cos\chi\end{Bmatrix}d\nu \qquad r \leq 1 \quad \begin{Bmatrix}0 \leq \theta \leq \theta_o\\\chi_o \leq \chi \leq \pi\end{Bmatrix}.$$

(c-2)-When $r \geq 1+\epsilon$, $\epsilon \geq 0$ and $0 \leq \theta \leq \theta_o$ or $\chi_o \leq \chi \leq \pi$ we may interchange the order of integration in (5) with differentiation of any order with respect to θ or r, and obtain the uniform convergence of the resulting integral expression with respect to these variables when they are in the given region. The same may be said for the integral in (6) when r,θ and χ lie in the closed intervals $0 < \epsilon \leq r \leq 1-\epsilon$, $0 \leq \theta \leq \theta_o$, $\chi_o \leq \chi \leq \pi$ respectively*.

(c-3)—Since the contribution on the contours C^ρ_+ and C^ρ_-** to the integrals derived from (5) and (6) by passing under integral sign with $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial r}$ approaches zero uniformly for ϕ in $(-\pi/4 \leq \phi \leq \pi/4)$ and $(3\pi/4 \leq \varphi \leq 5\pi/4)$ we may evaluate the latter by residues. Thus in particular equation (7.2) page 30, and (7.3) page 31, may be operated on termwise by $\frac{\partial}{\partial\theta}$ to obtain convergent eigenseries representations for $\frac{\partial u}{\partial\theta}$ in regions I and II.*** The footnote below tells us that these series converge at $\theta = \theta_o^+ (\theta = \theta_o^-)$.

(d)—For $r > 1$ the integral representations of $\frac{\partial u}{\partial r}, \frac{\partial u}{\partial\theta}, \frac{\partial^2 u}{\partial r^2}, \frac{\partial^2 u}{\partial\theta^2}$, are continuous across $\theta = \theta_o$.

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* Thus in particular

$$\frac{\partial u}{\partial\theta}(r,\theta_o^-) = \frac{1}{2\pi i}\int_{C^-_{\pi/4}} r^{-\nu}A(\nu)\frac{\partial P_{\nu-1}(\cos\theta)}{\partial\theta}\bigg|_{\theta=\theta_o} d\nu$$

and

$$\frac{\partial u}{\partial\theta}(r,\theta_o^+) = \frac{1}{2\pi i}\int_{C^-_{\pi/4}} r^{-\nu}B(\nu)\frac{\partial P_{\nu-1}(\cos\chi)}{\partial\theta}\bigg|_{\chi=\chi_o} d\nu$$

** See Figure 2, page 43

*** See Figure 5, page 29

To prove the assertions (a) thru (c) it will be necessary to have at our disposal the asymptotic expressions of certain functions which recur frequently.

Let

(7) $$\nu - \frac{1}{2} = \rho e^{i\phi}$$

Then from the facts

(8) $$P_\nu^\zeta (\cos \omega) = \nu^\zeta \sqrt{\frac{2}{\pi\nu \sin\omega}} \cos\left[(\nu+\tfrac{1}{2})\omega - \frac{\pi}{4} + \frac{\zeta\pi}{2}\right] (1 + 0\ (\nu^{-3/2})).$$

for real values of ζ and for $|\nu| \ggg 1$, $|\nu| \ggg |\zeta|$, $|\arg\nu| \leq \pi$, $0 < \epsilon \leq \omega \leq \pi - \epsilon$, ([2] page

(9) $P_\nu(\cos\omega) = P_{-\nu-1}(\cos\omega)$ ([2] page 56),

(10) $\frac{d}{d\omega} P_\nu(\cos\omega) = (-\nu)(\nu+1)P_\nu^{-1}(\cos\omega)$, ([2] page 63),

and

(11) $\frac{dP_\nu^{-1}(x)}{dx} = -\frac{1}{\sqrt{1-x^2}} P_\nu(x) + \frac{x}{\sqrt{1-x^2}} P_\nu^{-1}(x)$, ([2] page 62),

where $x = \cos\theta$ we can conclude,

(12) $$P_{\nu-1}(\cos\omega) \sim \frac{e^{\pm\omega\rho\sin\phi}}{\rho^{1/2}} \qquad \rho \longrightarrow \infty \begin{cases} +: 0 \leq \phi \leq \pi \\ -: \pi \leq \phi \leq 2\pi \end{cases},$$

(13) $$\frac{dP_{\nu-1}(\cos\omega)}{d\omega} \sim \rho^{1/2} e^{\pm\omega\rho\sin\phi} \qquad \rho \longrightarrow \infty \begin{cases} +: 0 \leq \phi \leq \pi \\ -: \pi \leq \phi \leq 2\pi \end{cases},$$

(14) $$\frac{d^2P_{\nu-1}(\cos\omega)}{d\omega^2} \sim \rho^{3/2} e^{\pm\omega\rho\sin\phi} \qquad \rho \longrightarrow \infty \begin{cases} +: 0 \leq \phi \leq \pi \\ -: \pi \leq \phi \leq 2\pi \end{cases},$$

where in all cases we have left out the constants containing ω and the phase factors of modulus 1 depending on ρ and ϕ *.

Now we know from page (23) of the text that as $|\nu| \longrightarrow \infty$

(15) $$\begin{cases} K^+(\nu-1/2,\theta_0) \sim (\nu-1/2)^{-1/2} & \mathrm{Re}(\nu-1/2) \geq 0 \\ K^-(\nu-1/2,\theta_0) \sim (1/2-\nu)^{-1/2} & \mathrm{Re}(\nu-1/2) \leq 0 \end{cases}$$

---

* In the asymptotic forms to follow we shall adhere to the same practice.

Therefore

$$(16)\begin{cases} K^{+}(\rho e^{i\phi},\theta_0) \sim \rho^{-1/2} & \rho \longrightarrow \infty, \left(-\pi/2 \le \phi \le \pi/2\right) \\ K^{-}(\rho e^{i\phi},\theta_0) \sim \rho^{-1/2} & \rho \longrightarrow \infty \left(\pi/2 \le \phi \le 3\pi/2\right) \end{cases}$$

Also,

$$(17) \quad r^{-\nu} = e^{-\nu \log r} \sim e^{-\rho\cos\phi \log r}$$

so that for $r > 1$ $(r < 1)$ and $-\pi/2 < \phi < \pi/2$, $(\pi/2 < \phi < 3/2\pi)$, $r^{-\nu} \longrightarrow 0$ exponentially.

Finally,

$$(18) \ \sin \pi\nu = \cos \pi(\nu - 1/2) \sim e^{\pm\pi\rho\sin\phi} \quad \text{as } \rho \longrightarrow \infty \qquad \begin{matrix} + : 0 \le \phi \le \pi \\ - : -\pi \le \phi \le 0 \end{matrix} .$$

Now we know from equations (6.5) and (6.6) of the text, that

$$(19)\begin{cases} A^{+}(\nu) = \dfrac{2\pi u_0}{K^{-}(-1/2,\theta_0)} \quad \dfrac{P_{\nu-1}(\cos \chi_0)}{K^{+}(\nu-1/2,\theta_0)(\nu\sin\pi\nu)} \\ B^{+}(\nu) = \dfrac{2\pi u_0}{K^{-}(-1/2,\theta_0)} \quad \dfrac{P_{\nu-1}(\cos \theta_0)}{K^{+}(\nu-1/2,\theta_0)(\nu\sin\pi\nu)} \end{cases},$$

and from (6.7) and (6.8) that

$$(20)\begin{cases} A^{-}(\nu) = \dfrac{u_0}{K^{-}(-1/2,\theta_0)} \quad \dfrac{K^{-}(\nu-1/2,\theta_0)}{\nu P_{\nu-1}(\cos \theta_0)} \\ B^{-}(\nu) = \dfrac{u_0}{K^{-}(-1/2,\theta_0)} \quad \dfrac{K^{-}(\nu-1/2,\theta_0)}{\nu P_{\nu-1}(\cos \chi_0)} \end{cases},$$

where the "+" and "-" superscripts have been appended to the A's and B's to indicate the fact that their use will be confined to $Re(\nu-1/2) \le 0$ and $Re(\nu-1/2) \ge 0$, respectively.

Using (12), (16) and (18),

$$(21)^{*}\begin{cases} A^{+}(\nu) \sim \rho^{-1} e^{\mp\theta_0\rho\sin\phi} & \text{as } \rho \longrightarrow \infty \quad \begin{cases} - : 0 \le \phi \le \pi/2 \\ + : 0 \ge \phi \ge -\pi/2 \end{cases} \\ A^{-}(\nu) \sim \rho^{-1} e^{\mp\theta_0\rho\sin\phi} & \text{as } \rho \longrightarrow \infty \quad \begin{cases} - : \pi/2 \le \phi \le \pi \\ + : \pi \le \phi \le 3/2\pi \end{cases} \end{cases}$$

*Henceforth all our considerations will be confined to the θ interval $0 \le \theta \le \theta_0$ corresponding to the integrand $A(\nu)$. Since we put no restriction on θ_0 except that $0 < \theta_0 < \pi$, it is clear, on making the substitution $\theta = \pi - \theta'$ and removing the prime on θ' that we have proved the corresponding assertions for the integrals containing $B(\nu)$.

Call $f^+(f^-)$ the integrand of equation (1) for $Re(\nu-1/2)\geq 0$ $(Re(\nu-1/2)\leq 0$. Similarly $f_r^+(f_r^-)$, $f_{rr}^+(f_{rr}^-)$, $f_{\theta\theta}^+(f_{\theta\theta}^-)$ are the corresponding integrands obtained from (1) be differentiating under the integral sign.

Asymptotically, as $\rho \longrightarrow \infty$, using the formulae already developed we have

$$(22)\quad \begin{cases} f^+ \sim \dfrac{e^{-\rho\cos\phi\log r}}{\rho^{3/2}}\, e^{-(\theta_0-\theta)\rho|\sin\phi|} , \\ f^- \sim \dfrac{e^{-\rho\cos\phi\log r}}{\rho^{3/2}}\, e^{-(\theta_0-\theta)\rho|\sin\phi|} , \end{cases}$$

$$(23)\quad \begin{cases} f_r^+ \sim \dfrac{e^{-\rho\cos\phi\log r}}{\rho^{1/2}}\, e^{-(\theta_0-\theta)\rho|\sin\phi|} , \\ f_r^- \sim \dfrac{e^{-\rho\cos\phi\log r}}{\rho^{1/2}}\, e^{-(\theta_0-\theta)\rho|\sin\phi|} , \end{cases}$$

$$(24)\quad \begin{cases} f_\theta^+ \sim \dfrac{e^{-\rho\cos\phi\log r}}{\rho^{1/2}}\, e^{-(\theta_0-\theta)\rho|\sin\phi|} , \\ f_\theta^- \sim \dfrac{e^{-\rho\cos\phi\log r}}{\rho^{1/2}}\, e^{-(\theta_0-\theta)\rho|\sin\phi|} , \end{cases}$$

$$(25)\quad \begin{cases} f_{rr}^+ \sim \rho^{1/2} e^{-\rho\cos\phi\log r} e^{-(\theta_0-\theta)\rho|\sin\phi|} , \\ f_{rr}^- \sim \rho^{1/2} e^{-\rho\cos\phi\log r} e^{-(\theta_0-\theta)\rho|\sin\phi|} , \end{cases}$$

$$(26)\quad \begin{cases} f_{\theta\theta}^+ \sim \rho^{1/2} e^{-\rho\cos\phi\log r} e^{-(\theta_0-\theta)\rho|\sin\phi|} , \\ f_{\theta\theta}^- \sim \rho^{1/2} e^{-\rho\cos\phi\log r} e^{-(\theta_0-\theta)\rho|\sin\phi|} . \end{cases}$$

In all cases the plus and minus signs used as superscripts on the f's are to be used in conjunction with $r\geq 1$ and $r\leq 1$ respectively. The plus and minus signs on the f's, it should be emphasized, indicate the validity of the asymptotic formulae in the regions $Re(\nu-1/2)\geq 0$ and $Re(\nu-1/2)\leq 0$ respectively.

With (22) thru (26) it is now a trivial matter to verify (a-1)(a-2),(b),(c-1) (c-2) and (c-3).

As for (d), we have to prove the continuity of $\frac{\partial u}{\partial \theta}$, $\frac{\partial u}{\partial r}$, $\frac{\partial^2 u}{\partial \theta^2}$ and $\frac{\partial^2 u}{\partial r^2}$ across $\theta = \theta_0$, $r > 1$. First, from (a-1) we know that the integral representation of $u(r,\theta)$ is continuous for all $r > 0$ and all θ such that $0 \leq \theta \leq \theta_0$ and $\chi_0 \leq \chi \leq \pi$.

Furthermore,*

(27) $$\int_{C^+_{\pi/4}} r^{-\nu} \left\{ A(\nu) \frac{\partial P}{\partial\theta} \nu{-}1(\cos\theta)\Big|_{\theta=\theta_o} - B(\nu) \frac{\partial P_{\nu-1}(\cos\chi)}{\partial\theta}\Big|_{\chi=\chi_o} \right\} d\nu$$

$$= \int_{C^+_{\pi/4}} r^{-\nu} \left\{ \phi_+(\nu) \right\} d\nu = 0 \quad ,$$

since from equation (6.4), giving the explicit formula of $\phi_+(\nu)$, we know that $\phi_+(\nu)$ is analytic in $Re(\nu-1/2) \geq 0$ and has the algebraic growth $|\nu|^{-1/2}$ in this region. Finally we consider the difference between $\lim_{\theta \to \theta_o^+} \frac{\partial^2 u}{\partial\theta^2}$ and $\lim_{\theta \to \theta_o^-} \frac{\partial^2 u}{\partial\theta^2}$. These limits exist, as shown above, and the difference is given by the following expression:

(28) $$\int_{C^+_{\pi/4}} r^{-\nu} \left\{ A(\nu) \frac{\partial^2 P}{\partial\theta^2}\nu{-}1(\cos\theta)\Big|_{\theta=\theta_o} - B(\nu) \frac{\partial^2 P}{\partial\theta^2}\nu{-}1(\cos\chi)\Big|_{\chi=\chi_o} \right\} d\nu \, .$$

But in general, from the Legendre differential equation,

(29) $$\frac{\partial^2 P_{\nu-1}(\cos\theta)}{\partial\theta^2} = \text{ctn}\,\theta \, \frac{\partial P_{\nu-1}(\cos\theta)}{\partial\theta} + (\nu)\cdot(\nu-1)P_{\nu-1}(\cos\theta)$$

Because of (27) above we need only show, therefore that

(30) $$\int_{C^+_{\pi/4}} r^{-\nu}(\nu)(\nu-1) \left\{ A(\nu)P_{\nu-1}(\cos\theta_o) - B(\nu)P_{\nu-1}(\cos\chi_o) \right\} = 0$$

But

(31) $$A(\nu)P_{\nu-1}(\cos\theta_o) = B(\nu)P_{\nu-1}(\cos\chi_o)$$

by equation (19) of this note, so that (30) is valid. The fact that

$$\int_{C^+_{\pi/4}} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial r^2} \end{pmatrix} r^{-\nu} \left\{ A(\nu)P_{\nu-1}(\cos\theta_o) - B(\nu)P_{\nu-1}(\cos\chi_o) \right\} = 0$$

follows also in the same manner from (31) so that finally (d) is completely proved.

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* It is hoped there will be no confusion caused by the use of ϕ for the argument of ν and $\phi(\nu)$ as the function of ν given in equation (3.20).

Note 4: <u>On the Summation of the Series Expansion of the Potential in the Case of the Disc.</u>

On page 30 equation (7.3) we showed that for the special case of $\theta_o = \pi/2$, that is when the cone becomes a disc, that

$$\text{a)}\quad u(r,\theta) = u_o - \frac{2u_o}{\pi}\sum_{m=0}^{\infty}(-1)^m \frac{r^{2m+1}}{2m+1} P_{2m+1}(\cos\theta)\,, \qquad r < 1\,.$$

The purpose of this note is to sketch the procedure used to sum the series in a) and to show that this sum is given by

$$\text{b)}\qquad u(r,\theta) = \frac{2u_o}{\pi}\tan^{-1}\left\{\sqrt{\frac{2}{r^2-1+\sqrt{(r^2-1)^2+4r^2\cos^2\theta}}}\right\}$$

(see equation 7.10, page 33), which is the result given by Smythe,[5], page 112.

First of all it is clear that equation a) may be written as

$$\text{c)}\qquad u(r,\theta) = u_o - \frac{2u_o}{\pi i}\sum_{p=0}^{\infty} i^{2p+1} P_{2p+1}(\cos\theta)\int_0^r t^{2p}\,dt$$

$$= u_o - \frac{2u_o}{\pi i}\int_0^r \frac{1}{t}\left\{\sum_{p=0}^{\infty} P_{2p+1}(\cos\theta)\,(it)^{2p+1}\right\} dt\,.$$

Now

$$\frac{1}{\sqrt{1-r^2-2\,ir\cos\theta}} = \sum_{p=0}^{\infty} P_{2p}(\cos\theta)(ir)^{2p} + \sum_{p=0}^{\infty} P_{2p+1}(\cos\theta)\,(ir)^{2p+1}$$

Thus

$$\text{d)}\quad \frac{1}{\sqrt{1-r^2-2ir\cos\theta}} - \frac{1}{\sqrt{1-r^2+2ir\cos\theta}} = 2\sum_{p=0}^{\infty} P_{2p+1}(\cos\theta)(ir)^{2p+1}\,.$$

Therefore

$$u(r,\theta) - u_o = \frac{u_o}{\pi i}\int_0^r \frac{1}{t}\left\{\frac{1}{\sqrt{1-t^2-2it\cos\theta}} - \frac{1}{\sqrt{1-t^2+2it\cos\theta)}}\right\} dt$$

$$= \frac{u_o}{\pi i}\lim_{\epsilon\to 0}\left\{\int_\epsilon^r \frac{dt}{t\sqrt{1-t^2-2it\cos\theta}} - \int_\epsilon^r \frac{dt}{t\sqrt{1-t^2+2it\cos\theta}}\right\}$$

Using formula 195, page 27 of Pierce's Tables we get

$$\text{e)} \int_\epsilon^r \frac{dt}{t\sqrt{1 \pm 2it\cos\theta - t^2}} = \log\left\{ \frac{2 \pm 2i\cos\theta\, t - 2\sqrt{1 \pm 2it\cos\theta - t^2}}{t} \right\} \Bigg|_\epsilon^r$$

Thus it is easily seen that

$$\text{f)}\ u(r,\theta) = u_o + \frac{u_o}{\pi i}\log\left\{ \frac{1 + ir\cos\theta - \sqrt{1 + 2ir\cos\theta - r^2}}{1 - ir\cos\theta - \sqrt{1 - 2ir\cos\theta - r^2}} \right\}$$

Note that the denominator, in the log of the above expression is the complex conjugate of the numerator.

Now

$$\tan^{-1} w = -\ \frac{i}{2}\log\left\{\frac{1 + iw}{1 - iw}\right\},$$

On putting the braced expression of equation f) into the form of the braced expression immediately above it is easy to see that

$$\text{g)}\quad w = \frac{r\cos\theta - \sqrt{\dfrac{\sqrt{} - (1-r^2)}{2}}}{1 - \sqrt{\dfrac{\sqrt{} + (1-r^2)}{2}}}$$

where,

$$\sqrt{} \equiv \sqrt{(1-r^2)^2 + 4r^2\cos^2\theta}\quad .$$

Thus we may write

$$\text{h)}\ u(r,\theta) = u_o + \frac{2u_o}{\pi}\tan^{-1}\left\{ \frac{r\cos\theta - \sqrt{\dfrac{\sqrt{} - (1-r^2)}{2}}}{1 - \sqrt{\dfrac{\sqrt{} + (1-r^2)}{2}}} \right\}\quad .$$

Returning for the moment to equation b), since,

$$\tan^{-1}\theta = \frac{\pi}{2} - \tan^{-1}\frac{1}{\theta}\ , \qquad \text{(Pierce 645)}$$

we have from b)

$$\text{i)}\ u(r,\theta) = u_o - \frac{2u_o}{\pi}\tan^{-1}\left\{\sqrt{\frac{(r^2-1) + \sqrt{}}{2}}\right\}\quad .$$

Now in order to show that the expressions for $u(r,\theta)$ given in h) and i) are equivalent we need only show that

$$\tan^{-1}\left\{\sqrt{\frac{(r^2-1)+\sqrt{}}{2}}\right\}+\tan^{-1}\left(\frac{r\cos\theta-\sqrt{\frac{\sqrt{}-(1-r^2)}{2}}}{1-\sqrt{\frac{\sqrt{}+(1-r^2)}{2}}}\right)=0\ .$$

Now

j) $$\tan^{-1}u+\tan^{-1}v=\tan^{-1}\left[\frac{u+v}{1-uv}\right]\ .$$

Thus if we show that $u+v=0$, we have proved our assertion. But this is easily verified so we have accomplished our purpose.

For $r>1$, $\theta_o=\frac{\pi}{2}$ an analogous procedure may be used in connection with equation (7.9) page 33.

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